Examination papers and Examiners' reports

Mathematics 2

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Undergraduate study in
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THE LONDON SCHOOL OF ECONOMICS AND POLITICAL SCIENCE

Examiner's report 2004

Zone A

Exam technique: general remarks

We start by emphasising that candidates should *always* include their working. This means two things. First, they should not simply write down the answer in the exam script, but explain the method by which it is obtained. Secondly, they should include rough working. The examiners want you to get the right answers, of course, but it is more important that you prove you know what you are doing: that is what is really being examined.

We also stress that if a candidate has not completely solved a problem, they may still be awarded marks for a partial, incomplete, or slightly wrong, solution; but, if they have written down a wrong answer and nothing else, no marks can be awarded.

Candidates should ensure that they have covered the bulk of the course in order to perform well in the examination: it is bad practice to concentrate only on a small range of major topics in the expectation that there will be lots of marks obtainable for questions on these topics. There are no formal options in this course: all students should study the full extent of the topics described in the syllabus and subject guide. In particular, since the whole syllabus is examinable, *any* topic could potentially appear in Section A.

Candidates are reminded that calculators are *not* permitted in the examination for this subject, under any circumstances. It is a good idea to prepare for this by attempting not to use your calculator as you study and revise this subject.

Specific comments on questions

1. We solve for the equilibrium quantity by solving

$$\frac{15}{q+3} + q + 1.$$

(Equally validly, we could solve for the equilibrium price rather than quantity.) This equation is equivalent to $(q + 3)(q + 1) = 15$ which, when written as a quadratic in standard form, is $q^2 + 4q - 12 = 0$, with positive (and hence economically meaningful solution $q = 2$. The equilibrium price is therefore $q^* = 2$ and the equilibrium quantity is $p^* = 3$.

The consumer surplus is

$$\int_0^{q^*} \frac{15}{q+3}\, dq - p^* q^* = \int_0^2 \frac{15}{q+3}\, dq - 6$$

$$= [15\ln(q+3)]_0^2 - 6 = 15\ln 5 - 15\ln 3 - 6,$$

which is the required answer.

A sketch graph reveals (since the supply equation is a straight line) that the producer surplus is the area of a triangular region with base 2 and height 2, and is therefore equal to 2, Alternatively (though this is more difficult), the formula

$$p^* q^* - \int_0^2 (q+1)\, dq = 6 - \left[q^2/2 + q\right]_0^2$$

may be used.

2. To check homogeneity, we need to verify that $f(cx, cy) = c^d f(x, y)$ for some d (which will be the degree of homogeneity). Now,

$$f(cx, cy) = cx \sin\left(\frac{cx}{cy}\right) + cxe^{-cy/cx} = c\left(x \sin\left(\frac{x}{y}\right) + xe^{-y/x}\right) = cf,$$

so f is homogeneous of degree 1. The partial derivatives are

$$\frac{\partial f}{\partial x} = \sin\left(\frac{x}{y}\right) + \frac{x}{y}\cos\left(\frac{x}{y}\right) + e^{-y/x} + \frac{y}{x}e^{-y/x},$$

$$\frac{\partial f}{\partial y} = -\frac{x^2}{y^2}\cos\left(\frac{x}{y}\right) - e^{-y/x}.$$

Euler's equation in this case is

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 1 \times f = f.$$

It is quite straightforward (assuming you have calculated the derivatives correctly!) to show that the left hand side of this equation does indeed simplify to f, which verifies Euler's equation.

3. The auxiliary equation is $m^2 + m - 12 = 0$ or $(m+4)(m-3) = 0$ yielding the complementary function $Ae^{-4x} + Be^{3x}$. A linear solution $y = Kx + L$ is sought and this gives on substitution

$$-12Kx + (K - 12L) = -12x - 11.$$

Hence, comparison gives $K = 1$ and $K - 12L = -11$, so that $L = 1$. We now seek a function $f(x) = x + 1 + Ae^{-4x} + Be^{3x}$ such that $f(0) = 2$, $f'(0) = 11$ Thus $2 = A + B + 1$ and $11 = 1 - 4A + 3B$.Hence $B = 2$ and $A = -1$. Thus the solution is

$$f(x) = x + 1 - e^{-4x} + 2e^{3x}$$

4. This question is best approached using elementary row operations. Cramer's rule is of limited use here. The augmented matrix corresponding to the system is

$$\begin{pmatrix} 2 & 1 & 1 & 4 \\ 6 & -1 & -1 & 4 \\ -4 & k & 6 & 8 \end{pmatrix}$$

and this may be reduced to echelon (row-reduced) form as follows:

$$\begin{pmatrix} 2 & 1 & 1 & 4 \\ 6 & -1 & -1 & 4 \\ -4 & k & 6 & 8 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 1 & 4 \\ 0 & -4 & -4 & -8 \\ 0 & k+2 & 8 & 16 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 6-k & 12-2k \end{pmatrix}.$$

The last row tells us that $(6-k)z = 2(6-k)$. This means that *if* $k = 6$ then the last equation vanishes and we have two consistent equations in three variables; there are then infinitely many solutions. If $k = 6$ we solve by letting $z = t$ with t arbitrary. From the second row we obtain $y = 2 - t$ and from the first row that $x = 1$. If k is *not* 6 then $(6-k)z = 2(6-k)$ implies that $z = 2$. hence t is no longer arbitrary; we have the unique solution $x = 1, y = 0, z = 2$. Note that

$$\begin{pmatrix} 2 & 1 & 1 \\ 6 & -1 & -1 \\ -4 & k & 6 \end{pmatrix}$$

has determinant equal to $8(k-6)$. If k is *not* 6 then Cramer's rule gives the unique solution $x = 1, y = 0, z = 2$. If, however, $k = 6$ then Cramer's rule is not applicable and row operations must be used.

5. There was quite a lot to do in this question, but it was all standard. (Incidentally, this question carries more credit than some other Section A questions. You should not assume that all questions carry the same credit. They are all compulsory in any case.) First, we need to find eigenvalues and eigenvectors of the matrix A. The characteristic polynomial is

$$p(\lambda) = \begin{vmatrix} 5-\lambda & -1 \\ -3 & 7-\lambda \end{vmatrix} = \lambda^2 - 12\lambda + 32 = (\lambda - 8)(\lambda - 4).$$

The eigenvalues are therefore 4 and 8. To find an eigenvector corresponding to $\lambda = 4$, we must find a solution (other than the zero vector) to

$$\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This amounts to the single equation $x - y = 0$. Taking $x = 1$ (for instance—any other non-zero choice will do), we obtain the eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. In a similar way, an eigenvector for $\lambda = 8$ is $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$, or any non-zero multiple of this. Now, if we take P to be

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix},$$

then $P^{-1}AP = D$ where

$$D = \begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix}.$$

For the next part, the question explicitly asks ("Use your result...") that we use the result just obtained. So an alternative approach would not have been correct. The system of differential equations can be written as

$$\begin{pmatrix} df/dt \\ dg/dt \end{pmatrix} = A \begin{pmatrix} f \\ g \end{pmatrix}.$$

Following the approach outlined in the subject guide, let's introduce two related functions F, G by $\begin{pmatrix} f \\ g \end{pmatrix} = P \begin{pmatrix} F \\ G \end{pmatrix}$. Then, the system of equations is equivalent to

$$P \begin{pmatrix} dF/dt \\ dG/dt \end{pmatrix} = A \begin{pmatrix} f \\ g \end{pmatrix} = AP \begin{pmatrix} F \\ G \end{pmatrix}.$$

So,

$$\begin{pmatrix} dF/dt \\ dG/dt \end{pmatrix} = P^{-1}AP \begin{pmatrix} F \\ G \end{pmatrix} = D \begin{pmatrix} F \\ G \end{pmatrix}.$$

So,

$$dF/dt = 4F, \quad dG/dt = 8G$$

and hence, for some numbers A and B, $F = Ae^{4t}$ and $G = Be^{8t}$. We're interested in determining f and g. We have

$$\begin{pmatrix} f \\ g \end{pmatrix} = P \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} Ae^{4t} + Be^{8t} \\ Ae^{4t} - 3Be^{8t} \end{pmatrix}.$$

Since $f(0) = 1$ and $g(0) = -1$, we must have $A + B = 1$ and $A - 3B = -1$, so $A = 1/2$, and $B = 1/2$. Finally, then, we have the solution:

$$f = \frac{1}{2}\left(e^{4t} + e^{8t}\right), \quad y_t = \frac{1}{2}\left(e^{4t} - 3e^{8t}\right).$$

6. This is a separable equation. Separating and integrating, we have

$$\int \frac{dy}{y} = \int \frac{x}{1+x^2}\, dx,$$

so

$$\ln y = \frac{1}{3} \ln(1 + x^3) + c$$

and, taking exponentials of each side, $y = A(1 + x^3)^{1/3}$ for some constant A. Now, $y(0) = 1$ means $1 = A(1)^{1/3}$, so $A = 1$ and $y = (1 + x^3)^{1/3}$ is the solution.

7. (a) There are various approaches. One is to directly use Taylor's theorem, calculating $f'(0), f''(0), f^{(3)}(0)$, $f^{(4)}(0)$ and $f^{(5)}(0)$ and using

$$f(x) \simeq f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4 + \frac{f^{(5)}(0)}{120}x^5.$$

These successive derivatives do, however, get harder to compute. An easier approach is to use the standard result (which you may assume) concerning the series for $\ln(1-y)$, which is

$$\ln(1-y) \simeq -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \frac{y^5}{5} - \frac{y^6}{6} \cdots .$$

Taking $y = 2x$, we obtain

$$\begin{aligned}\frac{\ln(1-2x)}{x} &\simeq \frac{1}{x}\left(-2x - \frac{(2x)^2}{2} - \frac{(2x)^3}{3} - \frac{(2x)^4}{4} - \frac{(2x)^5}{5} - \frac{(2x)^6}{6} \cdots\right) \\ &= -2 - 2x - \frac{8}{3}x^2 - 4x^3 - \frac{32}{5}x^4 - \frac{32}{3}x^5 \cdots .\end{aligned}$$

(b) To find the equilibrium quantity, we solve

$$\frac{A}{q+3} = q + 1.$$

(We could, instead, solve for the equilibrium price by solving $(A/p) - 3 = p - 1$. That would be equally correct.) This is equivalent to $(q+1)(q+3) = A$ which, in standard quadratic equation form, is

$$q^2 + 4q + (3 - A) = 0.$$

There are two solutions to this, and the economically meaningful (that is, non-negative) one is $q^* = -2 + \sqrt{1+A}$, and this is the equilibrium quantity. (We know $A > 3$, so this is non-negative.) The equilibrium price is therefore $p^* = (-2 + \sqrt{1+A}) + 1 = -1 + \sqrt{1+A}$. The consumer surplus is

$$\int_0^{q^*} \frac{A}{q+3}\, dq - p^*q^* = \int_0^{-2+\sqrt{1+A}} \frac{A}{q+3}\, dq - (-1+\sqrt{1+A})(-2+\sqrt{1+A})$$

$$= [A\ln(q+3)]_0^{-2+\sqrt{1+A}} - (-1+\sqrt{1+A})(-2+\sqrt{1+A}),$$

which turns out to equal the expression given in the question.

8. (a) There are at least two distinct approaches to finding the inverse: one is to use cofactors, and the other is by row operations. See the Subject Guide for details. The inverse of this matrix turns out to be

$$\begin{pmatrix} 2 & -1 & -1 \\ -2 & 1 & 2 \\ 3 & -1 & -2 \end{pmatrix}.$$

(b) The equilibrium price and quantity are easily seen to be, respectively, 12 and 14. With the imposition of an excise tax, the new equilibrium price p^T is given by

$$2(p^T - T) - 10 = 26 - p^T.$$

Make sure you understand why (read the Subject Guide and texts). Solving,

$$p^T = 12 + \frac{2}{3}T.$$

Then, the new equilibrium quantity is

$$q^T = 26 - p^T = 14 - \frac{2}{3}T.$$

With a tax of $R\%$, writing $r = R/100$, we have an equilibrium price p^R given by

$$2(1-r)p^R - 10 = 26 - p^R.$$

Again, make sure you understand why. Solving,

$$p^R = \frac{36}{3-2r} = \frac{36}{3-0.02R}.$$

Then, the corresponding quantity is

$$q^R = 26 - p^R = \frac{42-52r}{3-2r} = \frac{42-0.52R}{3-0.02R}.$$

If the two prices are equal, we have $p^R = p^T$, which means that

$$\frac{36}{3-2r} = 12 + \frac{2}{3}T,$$

leading to

$$r = \frac{3T}{36+2T},$$

so

$$R = 100r = \frac{300T}{36+2T}.$$

9. (a) The Lagrangian is

$$L(x,y,z,\lambda) = \left(\frac{1}{x}+1\right)\left(\frac{2}{y}+1\right)\left(\frac{3}{z}+1\right) - \lambda\left(x + \frac{y}{2} + \frac{z}{3} - c\right),$$

and the first order conditions are

$$\frac{\partial L}{\partial x} = -\frac{1}{x^2}\frac{(y+2)}{y}\frac{(z+3)}{z} - \lambda = 0,$$

$$\frac{\partial L}{\partial y} = -\frac{(x+1)}{x}\frac{2}{y^2}\frac{(z+3)}{z} - \frac{\lambda}{2} = 0,$$

$$\frac{\partial L}{\partial z} = -\frac{(x+1)}{x}\frac{(y+2)}{y}\frac{3}{z^2} - \frac{\lambda}{3} = 0,$$

$$x + \frac{y}{2} + \frac{z}{3} = c.$$

There are several ways to proceed. One is, for example, to note that the first three equations imply that

$$\frac{1}{x(1+x)} = \frac{4}{y(2+y)} = \frac{9}{z(3+z)}$$

Hence
$$x(1+x) = \frac{y}{2}\left(1+\frac{y}{2}\right) = \frac{z}{3}\left(1+\frac{z}{3}\right).$$
Thus $x + x^2 = \frac{y}{2} + \left(\frac{y}{2}\right)^2$ so that
$$\left(x - \frac{y}{2}\right)\left(1 + x + \frac{y}{2}\right) = 0.$$
But $x, y \geq 0$, so $x = y/2$ and similarly $x = z/3$. Hence $3x = C$ and so $x = C/3, y = 2C/3, z = C$. Finally, the minimum value is
$$f_{\min} = \left(\frac{1}{x}+1\right)\left(\frac{2}{y}+1\right)\left(\frac{3}{z}+1\right) = \left(\frac{3}{C}+1\right)^3.$$

(b) The difference equation has to be written in standard form before the usual techniques can be applied. This is
$$x_t + x_{t-1} - 2x_{t-2} = 0.$$
The auxiliary equation is $z^2 + z - 2 = 0$ and this has solutions -2 and 1. So, for some A, B,
$$x_t = A(-2)^t + B(1)^t = A(-2)^t + B.$$
Since $x_0 = 1$ and $x_1 = 2$, we have $A + B = 1$ and $-2A + B = 2$. So, $A = -1/3$ and $B = 4/3$ and hence
$$x_t = \frac{4}{3} - \frac{1}{3}(-2)^t.$$

10. (a) We have the difference equation
$$y_t = (1+r)y_{t-1} + I,$$
with initial condition $y_0 = D$. The time-independent solution is $y^* = I/(1-(1+r)) = -I/r$ and hence
$$y_t = -\frac{I}{r} + \left(D + \frac{I}{r}\right)(1+r)^t.$$

The account balance will be at least B after N years if $y_N \geq B$. That is, we need
$$-\frac{I}{r} + \left(D + \frac{I}{r}\right)(1+r)^N \geq B.$$
Solving this inequality gives
$$N \geq \frac{\ln\left((rB+I)/(rD+I)\right)}{\ln(1+r)},$$
as required.

(b) This is a linear equation, so an integrating factor should be found. This is

$$\mu = e^{\int (x/x^2+1)dx} = e^{\ln(x^2+1)/2} = \sqrt{x^2+1}.$$

So, we then have

$$y\sqrt{x^2+1} = \int \sqrt{x^2+1}x^3.$$

The integral can be evaluated using the substitution $u = x^2 + 1$, for instance, and we obtain

$$y\sqrt{x^2+1} = \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + c,$$

for some constant c. Given $y(0) = 1$, we have $1 = 1/5 - 1/3 + c$, so $c = 17/15$. Finally, then,

$$y = \frac{1}{5}(x^2+1)^2 - \frac{1}{3}(x^2+1) + \frac{17/15}{\sqrt{x^2+1}}.$$

Examination paper for 2005

The format of the 2005 examination will be the same as that of the 2004 examination; namely six compulsory questions in Section A and two questions to be chosen from 4 in Section B.