04b Sample Examination Problems Chapter 3 SOLUTIONS

1. A company which manufactures drink dispensing machines sets the fill level at 198cc. The standard deviation is 4cc. Assuming that the fill levels have a normal distribution,

   (a) What proportion of drinks will have less than 195cc?

   (b) What is the probability that a random sample of 50 drinks has a mean value greater than 199cc?

   (c) The company claims that an average drink is 200cc. What percentage of the sample means is 200cc or more if samples of size 36 are taken?

   (d) Explain briefly why you would or would not buy this dispensing machine.

(a) $X \sim N(198, 4^2)$, standardize : $Z = \frac{X - \mu}{\sigma} \sim N(0,1)$

$P(X < 195) = P\left(\frac{X - 198}{4} < \frac{195 - 198}{4}\right) = P(Z < -0.75)$

$= P(Z > 0.75) = 1 - P(Z < 0.75) = 0.2266$

22.66% of drinks have less than 195cc

(b) $P(\bar{X} > 199)$, n=50, $\bar{X} \sim N(\mu, \frac{\sigma^2}{n}) \sim N(198, 16/50)$

$= P\left(\frac{\bar{X} - 198}{\sqrt{16/50}} > \frac{199 - 198}{\sqrt{16/50}}\right) = P(Z > 1.77) = 1 - P(Z < 1.77) = 0.0384$

(c) n = 36 , $\bar{X} \sim N(\mu, \frac{\sigma^2}{n}) \sim N(198, 16/36)$

$P(\bar{X} > 200) = P\left(\frac{\bar{X} - 198}{\sqrt{16/36}} > \frac{200 - 198}{\sqrt{16/36}}\right) = P(Z > 3)$

$= 1 - P(Z < 3) = 0.00135$

0.13% is 200cc or more from a sample of size 36.

(d) No , there is 100 – 0.13 = 99.67% chance the claim of having an average of 200cc will not be met.

2. Suppose that X has a Poisson distribution with mean λ.
   (a) Find by summation the mean of X.
   (b) Find also the variance of X.

(a) **X ~ Pois(λ) (discrete distribution, i.e. use summation not integration)** $\Rightarrow$ **P(X = x)=** $p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}$

for $p_X(x)$ to be valid ,

$$p_X(x) \geq 0 \text{ and } \sum_{x=0}^{\infty} p_X(x) = 1 \text{ i.e. } \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = 1$$

$$E(X) = \sum_{x=0}^{\infty} x p_X(x) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$$

It is the same as starting from x = 1 ,since for x = 0 ,

The first term = (0)($\frac{e^{-\lambda}\lambda^x}{x!}$ **) = 0**

= $\sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$ **=** $\lambda \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^{x-1}}{x!}$ **,**

$$\frac{x}{x!} = \frac{x}{1.2.3......(x-1)(x)} = \frac{1}{1.2.3........(x-1)} = \frac{1}{(x-1)!}$$

= $\lambda \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}$ **the summation is X – 1 ~ Pois(λ)**

Therefore , $\sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} = 1$ **and hence** $E(X) = \lambda(1) = \lambda$

(b) Var (X) = E(X^2) – [E(X)]^2 = $\sum_{x=0}^{\infty} x^2 p_X(x) - \lambda^2$

Trick : E(X^2) = E[X(X-1)] + E(X)
Proof:E[X(X-1)]+ E(X)= E(X^2-X)+E(X)= E(X^2)-E(X)+E(X)= E(X^2)

E[X(X-1)]= $\sum_{x=0}^{\infty} x(x-1) p_X(x)$ **=** $\sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^x}{x!}$ **now for x = 0 and x = 1 the value of the terms = 0 , so we can start the summation from x = 2**

E[X(X-1)]= $\sum_{x=2}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^{x}}{x!}$, $\frac{x(x-1)}{x!}=\frac{1}{(x-2)!}$

$= \lambda^2 \sum_{x=2}^{\infty}\frac{e^{-\lambda}\lambda^{x-2}}{(x-2)!} = \lambda^2(1)=\lambda^2$

Now : Var (X) = E(X^2) – [E(X)]^2 = E[X(X-1)]+ E(X) - λ^2

$= \lambda^2 + \lambda - \lambda^2 = \lambda$

3. The distribution of random variable X has density function

$$f_X(x) = 1/3$$

where $-1 < x < 2$.

(a) Find by integration the mean of X.

(b) Find also the variance of X.

(c) What is the $P[X > 1 | X > 0]$?

(a) This is continous (uniform) distribution , to be valid :

(1) $f_X(x) \geq 0$ **and (2)** $\int_{-\infty}^{+\infty} f_X(x) = 1$

Although it is not required to verify that it is a valid pdf ,

(1) is satisfied as $f_X(x) = 1/3 \geq 0$ **for every x**

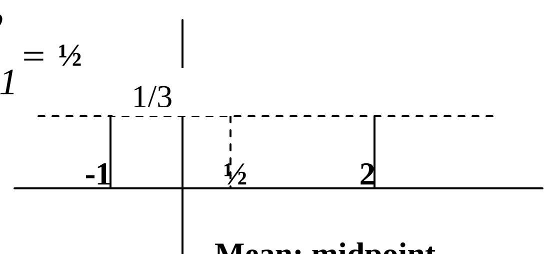
= 1/3 for -1 < x < 2 and equals 0 elsewhere.

(2) $\int_{-\infty}^{+\infty} f_X(x) = \int_{-\infty}^{-1} f_X(x) + \int_{-1}^{+2} f_X(x) + \int_{2}^{+\infty} f_X(x) = 0 + 1 + 0 = 1$

or from the graph = are of the rectangle = 3 (1/3) = 1

E(X) = $\int_{all} xf(x)dx = \int_{-1}^{2}\frac{x}{3}dx = \frac{x^2}{6}\Big|_{-1}^{2} =$ ½

You can see this from the graph:



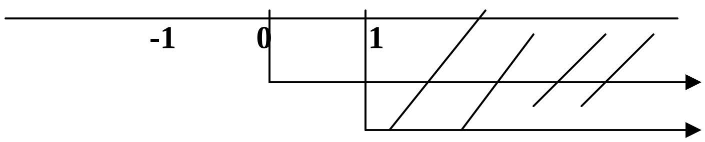
Mean: midpoint

(b) Var (X) = E(X²) – [E(X)]²

$$E(X^2) = \int_{all} x^2 f(x)dx = \int_{-1}^{2} \frac{x^2}{3} dx = \frac{x^3}{9}\Big|_{-1}^{2} = \mathbf{1}$$

$$\mathbf{Var\ (X) = E(X^2) - [E(X)]^2 = 1 - (1/2)^2 = 3/4}$$

(c) $P(X > 1 \mid X > 0) = \dfrac{P(X>1) \cap P(X>0)}{P(X>0)}$



The intersection is where the lines overlapp : X > 1

$$P(X > 1 \mid X > 0) = \frac{P(X>1) \cap P(X>0)}{P(X>0)} = \frac{P(X>1)}{P(X>0)} = \frac{\int_1^2 f_X(x)dx}{\int_0^2 f_X(x)dx}$$

$$= \frac{\int_1^2 1/3dx}{\int_0^2 1/3dx} = \frac{1/3}{2/3} = \frac{1}{2}$$

You may see it from the graph :

P(X > 1) = (1)(1/3) = 1/3 and P(X > 0) = (2)(1/3) = 2/3

4. If W is a Poisson random variable with mean 2, what is $P(W > 3 | W > 1)$?

W ~ Pois(2) $\Rightarrow P(W = w) = e^{-2} \dfrac{2^{-w}}{w!}$

$$\mathbf{P(W > 3 \mid W > 1)} = \frac{P(W>3) \cap P(W>1)}{P(W>1)} = \frac{P(W>3)}{P(W>1)} = \frac{1 - P(W \le 3)}{1 - P(W \le 1)}$$

$$P(W \le 3) = P(W = 0) + P(W = 1) + P(W = 2) + P(W = 3)$$

$$P(W \le 1) = P(W = 0) + P(W = 1)$$

$$P(W = 0) = e^{-2}\frac{2^0}{0!} = e^{-2} \quad , \quad P(W = 1) = e^{-2}\frac{2^1}{1!} = 2e^{-2}$$

$$P(W = 2) = e^{-2}\frac{2^2}{2!} = 2e^{-2} \quad , \quad P(W = 3) = e^{-2}\frac{2^3}{3!} = \frac{4}{3}e^{-2}$$

$$P(W > 3 \mid W > 1) = \frac{1-(19/3)e^{-2}}{1-3e^{-2}} = 0.2405$$

5. X is a random variable with $P(X = 0) = 0.1$, $P(X = 1) = 0.3$, $P(X = 2) = 0.4$. X can also take the value of 3, but no other values. What is $E[X^2]$?

X	**0**	**1**	**2**	**3**
P(X=x)	**0.1**	**0.3**	**0.4**	**c**

$$\sum_{x=0}^{3} p_X(x) = 1 \Rightarrow 0.1 + 0.3 + 0.4 + c = 1 \Rightarrow c = 0.2$$

$$E(X^2) = \sum_{x=0}^{3} x^2 p_X(x) = (0)^2(0.1) + (1)^2(0.3) + (2)^2(0.4) + (3)^2(0.2) = 3.7$$

6. If $x_1 = 3, x_2 = 2, x_3 = 4, x_4 = 2, x_5 = 5$, and all are equally likely values for X, what is $E[X(X-1)]$?

Equally likely : P(X = x) = $\frac{1}{5}$ **= 0.2**

X	**2**	**3**	**4**	**5**
P(X=x)	**0.4**	**0.2**	**0.2**	**0.2**

E[X(X-1)]= E(X²-X)= E(X²)-E(X) Or

E[X(X-1)]=

$$\sum_{x=2}^{5} x(x-1)p_X(x) = (2)(2-1)(0.4) + 6(0.3) + (12)(0.2) + (20)(0.2) = 8.4$$