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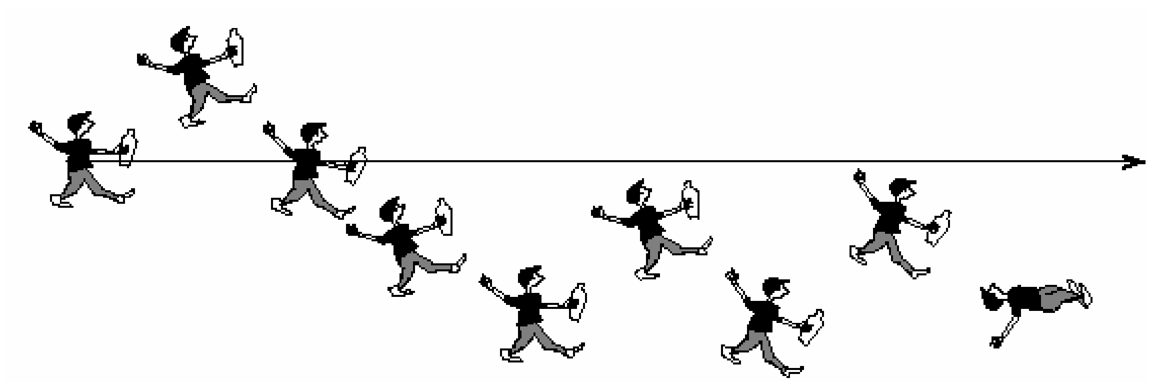
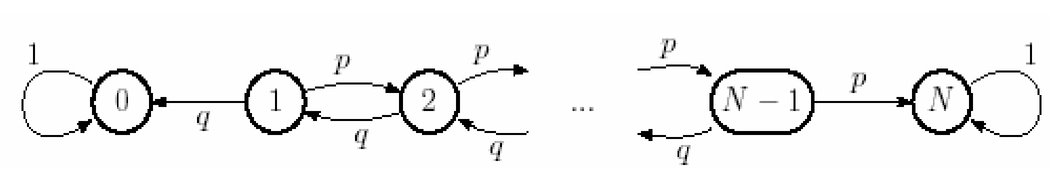
International Institute
for Technology and Management

Unit 76: Management Mathematics

Stochastic processes 4 Idiots





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Chapter 1: Introduction to Stochastic Processes

What's in this Chapter?

- **Quick Revision of Sample spaces and Random variables;**
- **Formal Definition of Stochastic Processes;**
- **Examples of Stochastic processes in Discrete time and Continuous time.**

A man walks into a bar, and before he can say a word, he is knocked unconscious. Why ? Answer[1]

Introduction:

You know that `wake-up' feeling? Your alarm goes off, you hit the `drowse' button, it goes off again 5 minutes later, you hit `drowse' again. . . Suppose that, each time your alarm sounds, you get out of bed with probability 0.4 and hit `drowse' with probability 0.6, independently of other occasions. This is an example of a stochastic process where you start at a **state** (sleep) and, at every step you either moves up (get out of bed) with a probability **p** or moves down (hit the button and sleep again) with a probability **1 – p** .

A ***Stochastic process*** can be described as a statistical phenomenon that evolves in time according to **probabilistic laws**. Probability provides models for analyzing random or unpredictable outcomes. The main new ingredient in stochastic processes is the explicit role of time. Knowledge of stochastic processes is essential for the study/analysis of computer networks, wireless communications, multimedia systems, financial market etc….

[1] The man had walked into an **iron** bar!

1.1 Revision: Sample spaces and Random variables

Definition: A **random experiment** is a physical situation whose outcome **can not be predicted until it is observed.**

Definition: A **Sample space**, **S**, is a set of **possible outcomes** of a random experiment.

Example: Random Experiment: Toss a coin once.
Sample Space: S= {Head, Tail}

Definition: A **random variable is** defined as a function from the sample space to the real numbers: **X:** $S \to \Re$

That is, a random variable **assigns** a real number to every possible outcome of a random experiment.

Example:

Random Experiment: Toss a coin once.

Sample Space: { Head , Tail }

Random Variable: X: $S \to \Re$

Maps "Head" $\to$ **1** ; "Tail" $\to$ **0**

Remember: A random variable is a way of producing random real numbers.

1.2 Formal Definition of Stochastic Processes

Definition: A **stochastic process** is a **family** of random variables **{ X(t) , t**$\in$**T }** where **t** usually denotes time.

That is, at every time **t** in the set **T**, a random number **X(t)** is observed. A stochastic process is described by its **position X(t)** at time **t** $\in$ **[0,1], t** $\in$ **[0,**∞ **) , or t** $\in$ **{0,1,2,3,…}.**

X(t) is called the State that the stochastic process can be in.
The state Space, S, is the set of values X(t) can take:
S = { X(t_1) , X(t_2), ……….}

Example: The easiest stochastic process to consider is the one made up of *independent* and *identically distributed* (**iid**) random variables.

Suppose we decided to play a game with a fair unbiased coin. We each start with $ 5 and repeatedly toss the coin.If turns up head ,then you give me a $1;If tails then I give you a $1. We continue until one of us has none and the other has $ 10. The sequence of heads and tails from the successive tosses would form a stochastic process.

Example: If **X(t)** is the outcome of a coin tossed at time t, then **X(t)** may take the values **0** or **1** ,i.e. X(t) = 0 or X(t) = 1

Hence **the State Space** is **S = {0, 1}**

1.3 Continuous Vs Discrete

Definition:{ X(t) , t∈T } is a **discrete time** process if the set **T** is finite or countable.

In practice, this means **T = {0, 1, 2, 3, 4 , …}**

Thus a discrete time process is **{ X(0) , X(1) , X(2) , …..}**

A new random variable is recorded at every time **0 , 1, 2 , 3 ,…**

Example: Observing the price of a share of a company at the beginning of each day:Beginning of day1,Beginning of day 2,.

Another example is the number of cars in a parking lot at the beginning of each hour.

Definition:{ X(t) , t∈T } is a **continuous time** process if the set **T** is **not** finite or countable.

In practice, this means **T = [0,1]**, or **T = [0,∞)**

In other words, a **continuous time** stochastic process is the process in which the **state** of the system can be **viewed at any time** ,not just at discrete instants of time.

Thus a continuous time process has a random number **X(t)** recorded at every *instant in time*.

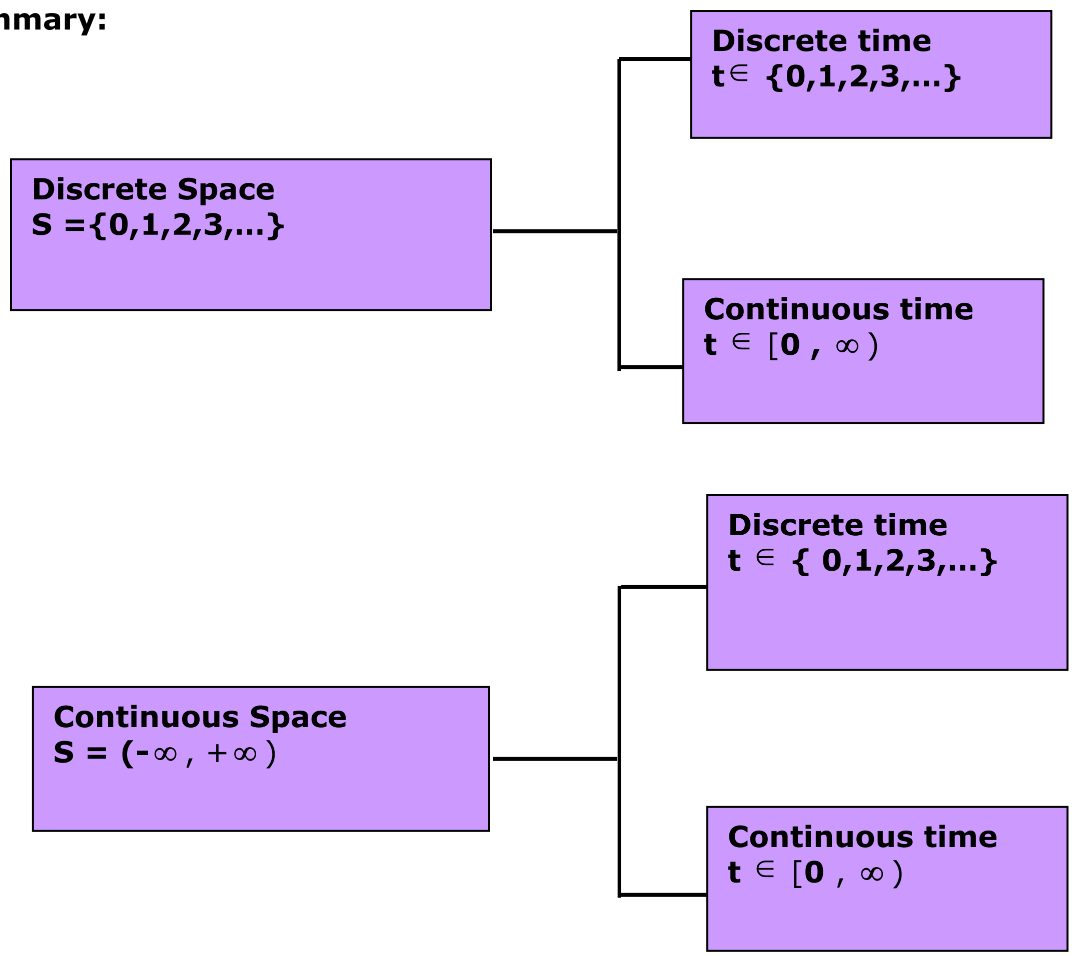
Example: The number of people in a supermarket **t** minutes after the store opens for business: number of people in the store in the first 5 minutes.(in [0,5]).

Definition: The state space **S** is **discrete** if it is **finite** or **countable** , otherwise it is **continuous**.

For discrete time we write X_t

For continuous time we write X(t)

Summary:



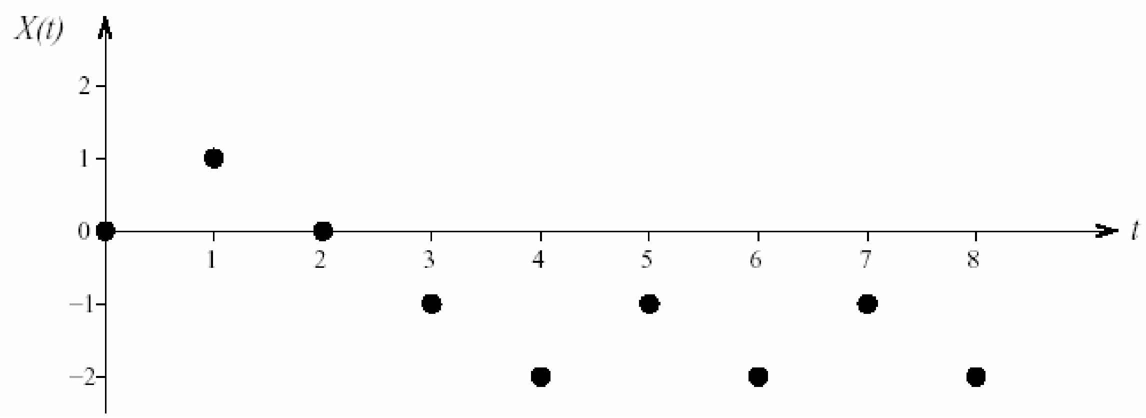
Examples:

1. **Random Walk on integers(Discrete space- Discrete time):**

   Let **T = { 0 ,1 ,2 , 3 ,……..}.** Start at **X(0) = 0** .

   At each time **1,2,3,….** ,the process takes a random step of size **1** up or down.

Fig1.1: Random walk on integers



That is, at time **1** it takes the step **Z_1 (-1 or +1)**,at time **2** it takes the step **Z_2 (-1 or +1) ,**etc…..

Denote the random step at time **i** by **Z_i ,** then :

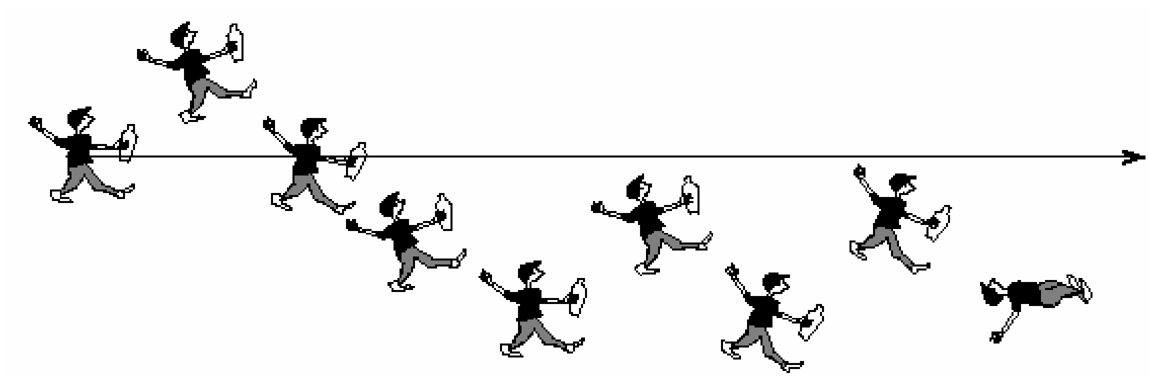
$$Z_i = \begin{cases} 1 & \text{with probability : p} \\ -1 & \text{with probability : 1 - p} \end{cases}$$

The state space is **S**={ …,-3,-2,-1,0,1,2,3,…} (set of Integers)

If **X_t** is the position at time **t** ,the random walk on integers **X_t** summarizes the outcome of all steps taken up to time **t** :

$$X_t = Z_1 + Z_2 + Z_3 + \ldots\ldots = \sum_{i=1}^{t} Z_i$$

The random walk on integers is often called the **drunkard's** walk because it resembles a drunk person walking along a street randomly to left or right.



Summary: Random walk on integers

Discrete space S={ …,-3,-2,-1,0,1,2,3,…}.

Discrete time T = { 0 ,1 ,2 , 3 ,……..}.

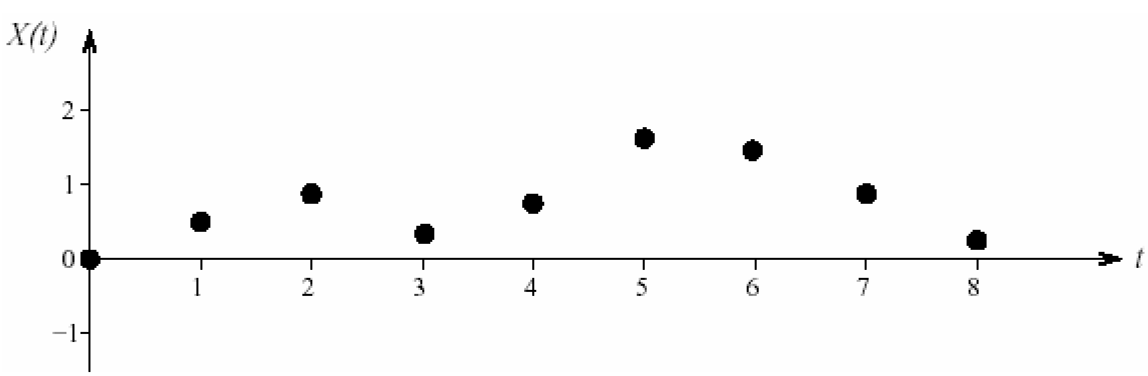
X_t is the position at time t

2. Random walk (Continuous space – Discrete time)

This time, we have **T = { 0,1,2,3,….}** as in Example1,but at every time **t** , a random step is taken of random size in **[-1,+1]** i.e. between **-1** and **+1 .**

(compare with the random walk on integers where each step had exactly a size of **-1** or **+1**).

Fig 1.2 : Random walk with continuous state space



The process looks similar to the Random walk on integers, but its movements up and down are more gentle because the step sizes are smaller.(compare with fig. 1.1).

Summary: Random walk –Continuous space

Discrete space S = $\Re$

Discrete time T = { 0 ,1 ,2 , 3 ,……..}.

X_t is the position at time t

3. Queue Length (Discrete space –Continuous time)

Consider a queue at a bank, new customers arrive in the queue at random instants. Customers also leave at random instants after being served. The queue length $\mathbf{X_t}$: the number of customers in the queue at time **t**.

$\mathbf{X_t}$ is a stochastic process with discrete state space **S = {0,1,2,….}** in continuous time **T = [0,∞)**.

Summary: Queue length.

Discrete state space,S ={0;1; 2; ……}.

Continuous time, T = [0; ∞).

X(t) = number in queue at time t.

3. Brownian Motion (Continuous space – Continuous time)

The term Brownian motion (in honor of the botanist Robert Brown) refers to either the physical phenomenon that minute particles immersed in a fluid move about randomly; or the mathematical models used to describe those random movements.

Brownian motion describes the motion of particles that are moving continuously and randomly in both space and time.

X(t) is the position in space of the particle at time t.

Summary: Brownian motion.

Continuous state space, S = R.

Continuous time, T = [0; ∞).

X(t) = (one-dimensional) position of particle at time t.

Exercises

For each of the following stochastic processes, give the state space and state whether the process is discrete-time or continuous-time:

1. The number of occupied channels in a telephone link at the arrival time of the 10^{th} call.
2. The number of occupied channels in a telephone link at time $t > 0$.
3. Number of defects in batches of items
4. Number of customers in line at a bank
5. Quantity of gas at a gas station at the end of each business day.
6. Height of snow on the top of Everest.
7. **Speeding tickets**. A traffic policeman spends 3 hours at the road-side issuing tickets to speeding drivers. Let X(t) be the number of tickets the policeman has issued by time t.
8. **Cat flea diffusion**. A lecturer has two cats, George and Tony, who have **10** fleas between them. Once a day, the cats stand side-by-side as they eat their breakfast, and fleas can hop from one cat to the other. Let **X(t)** be the number of fleas that George has on day **t**.
9. **Tennis rally**. Venus and Serena are playing a game of tennis. Let X(t) be the speed (in km/h) at which the ball travels immediately after hit t (where X(0) is the service speed).

Answers

1. State space: S = {0; 1; 2; ….. ;10} (discrete).
Discrete time: T = {0; 1; 2; 3; ……}

2. State space: S = {0; 1; 2; ………} (discrete).
Continuous time: T = (0; ∞)

3. State space: Discrete
Discrete time

4. State space: Discrete
Continuous time

5. State space: Continuous
Discrete time

6. State space: Continuous
Continuous time

7. State space: S = {0; 1; 2; ………} (discrete).
Continuous time: T = [0; 3] (in hours).

8. State space: S = {0; 1; 2; ….. ;10} (discrete).
Discrete time: T = {0; 1; 2; 3; ……} .

9. State space: S = [0;1) in km/h (continuous).
Discrete 'time': T = {0; 1; 2; 3; …….}
(t = 0 for serve, t = 1 for first hit, etc.)

Chapter 2: Markov Chains

NEXT depends on NOW ,not BEFORE

What's in this Chapter?

- **Definition of Markov Chains;**
- **Properties of Markov Chains;**
- **Examples on Markov Chains.**

Andre' Markov
1856 - 1922

Introduction

Markov chains are stochastic processes, where the outcomes at any stage depend upon the previous stage(and No Further back).

Example 2.1

Remember: In a Markov chain, the future depends only upon the present.

The previously mentioned game that we started with $ 5 each , illustrates a Markov chain:(see Example p. 5)

Consider the stochastic process formed by the number of dollars you have immediately after each toss of the coin.

The sequence would start at $**5** and increase or decrease by $**1** at each step until eventually the process would cease with a value of either $**0** or $ **10**.

Assume after several tosses, you currently have $3.The probability that after the next toss you will have $4 is 0.6 and the knowledge of the past (i.e. how many $'s you had one or two tosses ago) does not help in calculating this probability.

The **Future : How many $'s you will have after the next toss.**

is independent of

The **Past : How many $'s you had several tosses ago.**

Given The **Present : You currently have $ 3.**

Markov Property

The basic property of a Markov chain is that only the most recent point in the path affects what happens next.
This is called the Markov Property.
It means that the state at time t + 1 : $\mathbf{X_{t+1}}$, depends only on the state at time t : $\mathbf{X_t}$. It does not depend upon $\mathbf{X_0; X_1 ; \ldots\ldots ; X_{t-1}}$.

We formulate the Markov Property in mathematical notation as follows

$$P(Xt+1 = s \mid X_t = s_t ; X_{t-1} = s_{t-1} ;\ldots; X_1 = s_1; X_0 = s_0) = P(X_{t+1} = s \mid X_t = s_t)$$

for all t = 1; 2; 3; ……. and for all states s_0; s_1 ; ……… ; s_t ; s.

Explanation:

$P(X_{t+1}=s \mid X_t = s_t ; Xt-1 = st-1 ;\ldots; X_1 = s_1; X_0 = s_0) = P(X_{t+1} = s \mid X_t = s_t)$

Distribution of X_{t+1} — **Depends on X_t** — **whatever happened before time t doesn't matter.**

Example 2.2

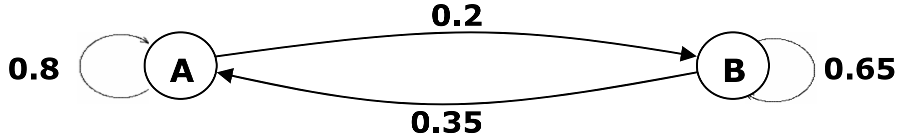
A small town has only two dry cleaners **A** and **B**. the manager of **A** hopes to increase the firm's market share by an extensive advertising campaign. After the campaign, a market research firm finds out that there is a probability of **0.8** that an **A** customer will use A's services and a **0.35** chance that a **B**'s customer will switch to **A**'s services.

If a customer bringing his load to A is said to be in **state 1** and a customer bringing load to **B** is said to be in **state 2**

If there is **0.8** chance that an **A** customer will come back to **A** ,then there is **1- 0.8 =0.2** chance that the customer will switch to **B**.

If there is **0.35** chance that a customer will switch to **A**, then there is **1 – 0.35 = 0.65** that a customer will return to **B**.

Fig. 2.1 shows the **State Transition Diagram ,** the numbers that appear as labels on the arrows are the **transition probabilities**.



We can also summarize the probabilities in a **transition matrix:**

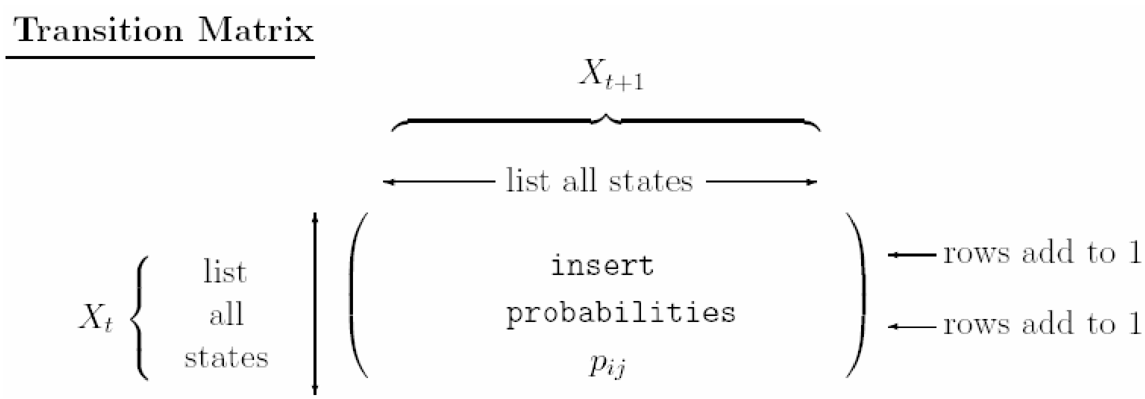
$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} 0.8 & 0.2 \\ 0.35 & 0.65 \end{pmatrix}$$

$\mathbf{p_{11}}$: Probability of going from state1 to state 1 = **0.8**

$\mathbf{p_{12}}$: Probability of going from state1 to state 2 = **0.2**

$\mathbf{p_{21}}$: Probability of going from state2 to state 1 = **0.35**

$\mathbf{p_{22}}$: Probability of going from state2 to state 2 = **0.65**



Properties of the Transition Matrix

1. It is a square matrix **(n x n)**
2. All entries are between ***0** and **1***, inclusive, $0 \le p_{ij} \le 1$ because all entries represent probabilities.
3. The sum of entries in any row must be **1** .
4. Entry (i;j) is the conditional probability that **NEXT= j** , given

   that **NOW= i** : $\mathbf{p_{ij} = P(X_{t+1} = j \mid X_t = i)}$

<u>Important</u>: The Meaning of the entries:

-Every entry represents moving from one state to another in **one step**.

- A step means a **time** ,if $p_{12} = 0.2$ then this means it is possible to go from state 1 to state 2 at time **t = 1** (in one step) with a probability of 0.2.
- Every entry represent a **direct path** , if $p_{21} = 0$,then this means it is **not** possible to go from state 2 to state 1 directly at **t = 1** .However it may be possible to go from state 2 to state 1 in more than one step at t = 2,3,etc.

Example 2.3

Consider the following transition matrix:

p_{21}:at t = 2 in state 1

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.9 & 0.1 & 0 \\ 0.1 & 0.7 & 0.2 \\ 0 & 0.5 & 0.5 \end{pmatrix} \end{array}$$

p_{32}:at t = 1 in state 2 **p_{33}: at t = 0 in state3**

There is no direct path from **3** to **1** ,i.e. at **t = 1** , it is not possible to go from **3** to **1** since **p_{31} = 0** .We would like to see if this is possible(**3**$\rightarrow$**1**) in more steps.

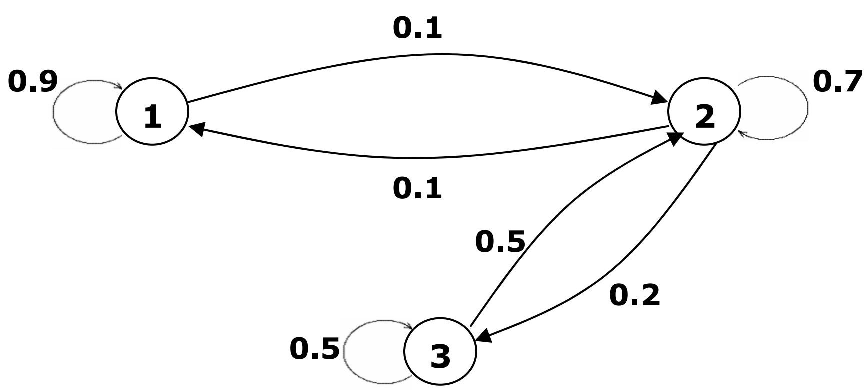
Suppose that the process is in state **3** at time **t = 0** .

First you look whether one can leave **3** : it is possible to go from **3** to **2** in the next step (**t = 1**) since **p_{32} = 0.5**

It is also possible to go from **2** to **1** in the step that follows (**t = 2**) since **p_{21} = 0.1**

Hence **1** cannot be reached from **3** directly (in one step at **t= 1**) ,but can be reached from **3** in two steps at **t = 2** .

This can be seen clearly with the state transition diagram :



time	t=0		t=1		t = 2	
state:	**3**	$\xrightarrow{0.5}$	**2**	$\xrightarrow{0.1}$	**1**	
	step0		step1		step2	

The probability of reaching state 1 from state 3 in two steps (**t = 2**)

= p_{32} x p_{21} = (0.5)(0.1) = 0.05

You may find this probability by squaring the matrix: $\mathbf{P^2}$

Entries of P^2 represent going from one state to another in 2 steps (this is discussed later in this guide):

$$\mathbf{P^2 = P.P} = \begin{pmatrix} 0.9 & 0.1 & 0 \\ 0.1 & 0.7 & 0.2 \\ 0 & 0.5 & 0.5 \end{pmatrix}\begin{pmatrix} 0.9 & 0.1 & 0 \\ 0.1 & 0.7 & 0.2 \\ 0 & 0.5 & 0.5 \end{pmatrix} = \begin{pmatrix} 0.82 & 0.16 & 0.02 \\ 0.16 & 0.6 & 0.24 \\ 0.05 & 0.6 & 0.35 \end{pmatrix}$$

In $\mathbf{P^2}$ **: $p_{31} = 0.05$ is the probability of going from 3 to 1 in 2 steps. Same result we've got above.**

Distribution Vectors

Suppose, in **Example 2.2**, that when the campaign began, **A** had **40%** of the market and **B** had **60%** we need to find how these proportions would change after another week of advertising.

All we need to do is to **multiply** the initial distribution **(0.4 0.6)** by the transition matrix :

$$\begin{pmatrix} \mathbf{0.4} & \mathbf{0.6} \end{pmatrix}\begin{pmatrix} \mathbf{0.8} & \mathbf{0.2} \\ \mathbf{0.35} & \mathbf{0.65} \end{pmatrix} = \begin{pmatrix} \mathbf{0.53} & \mathbf{0.47} \end{pmatrix}$$

Initial Distribution ↑ — **Transition Matrix** ↑ — **Distribution after 1 step** ↑

The initial distribution of 40% and 60% becomes after one week **(one step)** 53% and 47%.These distributions can be written as the ***probability vectors***: **[0.4 0.6] and [0.53 0.47]**

A ***probability vector*** is a matrix of only one row, having non-negative entries with the sum of entries equal to 1.

To find the market share after two weeks, multiply the vector (0.53 0.47) by the transition matrix:

$$\begin{pmatrix} 0.53 & 0.47 \end{pmatrix}\begin{pmatrix} 0.8 & 0.2 \\ 0.35 & 0.65 \end{pmatrix} = \begin{pmatrix} 0.59 & 0.41 \end{pmatrix}$$

Distribution after 1 step ↑ — **Transition Matrix** ↑ — **Distribution after 2 steps** ↑

The following table gives the market share (rounded) for each cleaner after various weeks:

Week	A	B
Start	0.4	0.6
1	0.53	0.47
2	0.59	0.41
3	0.62	0.38
4	0.63	0.37
5	0.63	0.37
…..	…..	…..
12	0.64	0.36

The results seem to approach the probability vector

(0.64 0.36). This vector is called the ***equilibrium vector*** or the fixed vector for the given transition matrix.

The ***equilibrium vector*** gives a **long-range-prediction-the shares** of the market will stabilize(under the same conditions)at **64%** for **A** and **36%** for **B**.

By definition, π is the *fixed probability* vector (*equilibrium probability vector*) if $\pi P = \pi$ where P is the transition matrix.

To find the equilibrium vector of the example above:

Let it be π $(\pi_1 \quad \pi_2)$ then:

$$\begin{pmatrix}\pi_1 & \pi_2\end{pmatrix}\begin{pmatrix}0.8 & 0.2\\ 0.35 & 0.65\end{pmatrix} = \begin{pmatrix}\pi_1 & \pi_2\end{pmatrix}$$

$$\begin{pmatrix}0.8\pi_1 + 0.35\pi_2 & 0.2\pi_1 + 0.65\pi_2\end{pmatrix} = \begin{pmatrix}\pi_1 & \pi_2\end{pmatrix}$$

$0.8\pi_1 + 0.35\pi_2 = \pi_1$

$0.2\pi_1 + 0.65\pi_2 = \pi_2$

Now these two equations tell you something about π_1 and π_2 but not enough because they are really the same equation(Do U see that?).

There is one more thing we know though, Since **(**$\pi_1 \quad \pi_2$**)** is a **probability vector**, its entries must add up to **1** :

π_1 **+** π_2 **= 1**

Using this equation with one of the above equations :

Remember:
Starting with some other initial probability vector would give the **same** equilibrium vector. In fact, the **long range trend is same no matter what the initial vector is. The long range trend depends only on the transition matrix not on the initial distribution.**

π_2 **= 0.364 ≈0.36** and π_1 **= 0.636 ≈ 0.64**
π **= (0.64 0.36)**

We call the probability vector π with the property $\pi P = \pi$ a **steady-state probability vector**. We call P^{∞} the **long-term** or **steady-state transition matrix.** The rows of the steady-state transition matrix are steady-state probability vectors.

Example 2.4

Obtain the equilibrium distribution of the process for the following transition matrix:

Let $\pi(\pi_{-2},\pi_{-1},\pi_0,\pi_1,\pi_2)$ be the equilibrium vector: $\pi P = \pi$

$$(\pi_{-2},\pi_{-1},\pi_0,\pi_1,\pi_2)\quad \begin{array}{c} \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccccc} \mathbf{-2} & \mathbf{-1} & \mathbf{0} & \mathbf{1} & \mathbf{2} \end{array} \\ \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix} \end{array} = (\pi_{-2},\pi_{-1},\pi_0,\pi_1,\pi_2)$$

$1/3\ \pi_{-1} + 1/3\ \pi_0 + 1/3\ \pi_1 = \pi_{-2}$

$½\ \pi_{-2} + ½\ \pi_0 = \pi_{-1}$

$½\ \pi_{-1} + ½\ \pi_1 = \pi_0$

$½\ \pi_0 + ½\ \pi_2 = \pi_1$

$1/3\ \pi_{-1} + 1/3\ \pi_0 + 1/3\ \pi_1 = \pi_2$

AND $\pi_{-2} + \pi_{-1} + \pi_0 + \pi_1 + \pi_2 = 1$

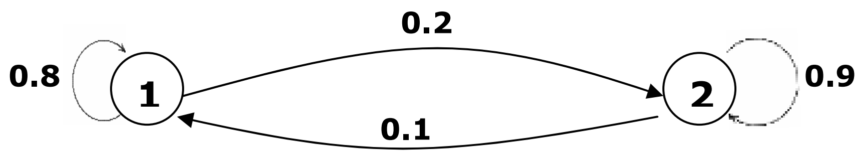
Symmetry :

The first and the fifth rows are identical : $\pi_{-2} = \pi_2$
The second and the fourth rows are symmetrical : $\pi_{-1} = \pi_1$

Solving : $\pi(\frac{3}{26},\frac{3}{13},\frac{4}{13},\frac{3}{13},\frac{3}{26})$

Example 2.5

The state transition diagram and the transition matrix of two products' exchange are given below :



(1) : Uses Product 1 **(2) : uses product 2**

P = $\begin{pmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{pmatrix}$ Calculate **P^2 , P^4 , P^8 , P^{16} and P^{32}**

P^2 = P . P = $\begin{pmatrix} 0.66 & 0.34 \\ 0.17 & 0.83 \end{pmatrix}$; **$P^4 = P^2 . P^2$ =** $\begin{pmatrix} 0.4934 & 0.5066 \\ 0.2533 & 0.7467 \end{pmatrix}$

$P^8 = P^4 . P^4$ = $\begin{pmatrix} 0.3718 & 0.6282 \\ 0.3141 & 0.6859 \end{pmatrix}$; **P^{16}=P8. P^8 =** $\begin{pmatrix} 0.3355 & 0.6645 \\ 0.3322 & 0.6678 \end{pmatrix}$

$P^{32} = P^{16}. P^{16}$ = $\begin{pmatrix} 0.3333 & 0.6667 \\ 0.3333 & 0.6667 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$

The higher and higher powers seem to be getting closer and closer to

The matrix $\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ this is what we call **long-term** or **steady-state**

transition matrix P^∞ ,note that the rows of P^∞ are the same.

Note also that the **steady-state probability vector** is $\pi \ (\frac{1}{3} \quad \frac{2}{3})$

Regular Markov Chains

Question: Is there always a steady state vector and a long-term transition matrix?

Steady-state vectors always exist ,but if we keep multiplying **P** by itself ,the answers need **not** to get closer and closer to the matrix whose rows are a steady-state vector.

For instance **P** = $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has π **(0.5 0.5)** as a steady-state vector, but if we take higher and higher powers of **P** , we find the odd powers of **P** are equal to **P**, while the even powers of P are equal to the Identity matrix. Thus the powers of P do not stabilize but hop back and forth between **P** and the identity matrix.

Question: Under what circumstances can we be sure that the higher powers of P will stabilize? In other words when P^{∞} exist?

When the Markov system is **Regular**.
A system with transition matrix **P** is said to be regular if there is Some power of **P** contains no *zero* entries.

Example 2.6

Determine whether the system with the given transition matrix is regular:

1. P = $\begin{pmatrix} 0.5 & 0 & 0.5 \\ 0.6 & 0 & 0.4 \\ 0 & 1 & 0 \end{pmatrix}$ **2. P =** $\begin{pmatrix} 0.6 & 0.4 \\ 0 & 1 \end{pmatrix}$

1. Here is a useful technique for determining whether a system is regular. Start by replacing non-zero entries in the transition matrix by **+ :**

$\begin{pmatrix} + & 0 & + \\ + & 0 & + \\ 0 & + & 0 \end{pmatrix}$ we can compute the product **P²** **symbolically:**

$$\mathbf{P^2} = \begin{pmatrix} + & 0 & + \\ + & 0 & + \\ 0 & + & 0 \end{pmatrix}\begin{pmatrix} + & 0 & + \\ + & 0 & + \\ 0 & + & 0 \end{pmatrix} = \begin{pmatrix} + & + & + \\ + & + & + \\ + & 0 & + \end{pmatrix}$$

For example, the 1,2 entry in the product :

(+.0)+(0.0) + (+.+) = +

$$\mathbf{P^4 = P^2.\,P^2} = \begin{pmatrix} + & + & + \\ + & + & + \\ + & 0 & + \end{pmatrix}\begin{pmatrix} + & + & + \\ + & + & + \\ + & 0 & + \end{pmatrix} = \begin{pmatrix} + & + & + \\ + & + & + \\ + & + & + \end{pmatrix}$$

Since $\mathbf{P^4}$ has all positive entries ,the system is regular.

2. The transition matrix has the symbolic form **:**

$$\mathbf{P} = \begin{pmatrix} + & + \\ 0 & + \end{pmatrix} \quad ; \mathbf{P^2} = \begin{pmatrix} + & + \\ 0 & + \end{pmatrix}\begin{pmatrix} + & + \\ 0 & + \end{pmatrix} = \begin{pmatrix} + & + \\ 0 & + \end{pmatrix}$$

You can continue multiplying powers of **P** forever, but a little thought will convince you that you will keep getting back the same symbolic form, with a zero in the 2,1 position.

Thus the system is not regular.

Powers of the transition Matrix

Question : What do the entries of P^2 mean ?

P is the **one step** transition matrix and $\mathbf{P^2}$ is the **two step** transition matrix. In other words, the $\mathbf{p_{12}}$ entry in $\mathbf{P^2}$ is the **probability of going from state1 to state 2 in two steps.**

Example 2.7

P = $\begin{pmatrix} 0 & 1 \\ 0.5 & 0.5 \end{pmatrix}$ **One step transition matrix**

$P^2 = P.P = \begin{pmatrix} 0.5 & 0.5 \\ 0.25 & 0.75 \end{pmatrix}$ **Two steps transition matrix**

The probability of going from state **1** to state **2** in **two** steps is p_{12} entry in P^2 **= 0.5**

$P^3 = P^2.P = \begin{pmatrix} 0.25 & 0.75 \\ 0.375 & 0.625 \end{pmatrix}$ **Three steps transition matrix**

The probability of going from state **1** to state **2** in **three** steps is p_{12} entry in P^3 **= 0.75**

Example 2.8

Consider a Markov chain with transition matrix

$$P = \begin{matrix} & 1 & 2 \\ 1 & 0.4 & 0.6 \\ 2 & 0.9 & 0.1 \end{matrix}$$

Assume that the chain starts from state **1** at time **t = 0** .

What is the probability that it is in state **1** at time **t= 4** ;

the entry p_{11} in P^4 is the probability that the system is in state **1** at time **t = 4** :

$$P^2 = P.P = \begin{pmatrix} 0.4 & 0.6 \\ 0.9 & 0.1 \end{pmatrix}\begin{pmatrix} 0.4 & 0.6 \\ 0.9 & 0.1 \end{pmatrix} = \begin{pmatrix} 0.7 & 0.3 \\ 0.45 & 0.55 \end{pmatrix}$$

$$P^4 = P^2.\, P^2 = \begin{pmatrix} \boxed{0.7 \quad 0.3} \\ 0.45 \quad 0.55 \end{pmatrix}\begin{pmatrix} \boxed{\begin{matrix}0.7 \\ 0.45\end{matrix}} & \begin{matrix}0.3 \\ 0.55\end{matrix} \end{pmatrix}$$

Since we need only p_{11} ,it is enough to multiply the first row with the first column : p_{11} **= (0.7)(0.7) + (0.3)(0.45) = 0.625**

Construction of the transition matrix

Drunkard's Walk

A man walks along a four-block stretch of an avenue. If he is at corner 1,2 or 3 then he walks to the left or right with equal probabilities ($p=q=\frac{1}{2}$) he continues until he reaches corner 4 which is a bar or corner 0 which is his home. If he reaches home or the bar he stays there. **States 0 and 4 are called *absorbing* states.**

Absorbing State

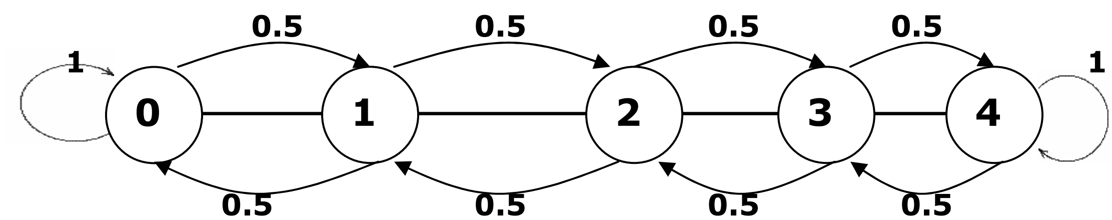
Definition: A state is ***absorbing*** if once the state is entered, it is impossible to leave it.

Remarks :

1. A state ***i*** is absorbing if $p_{ii} = 1$;in our example: $p_{00} = 1; p_{44} = 1$

2. Since rows add up to **1** , Absorbing states have a **1** and **0's** in their corresponding rows.

Fig. 2.2 State transition diagram with states 0,1,2,3,4



Absorbing Chain

Definition: A Markov chain is ***absorbing*** if it has at **least one** absorbing state and it is possible to get from **each** non-absorbing states in one or more time steps.

From the diagram, the states **1 , 2 and 3** are non-absorbing *(transient)* and from any of these , it is possible to reach the absorbing states **0** and **4**.Hence the chain is an **absorbing chain**.

The transition matrix : **5 states** $\rightarrow$ **5x5 matrix**

Absorbing states have a 1 and 0's in their corresponding rows.

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccccc} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} = \begin{array}{c} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{array}\begin{array}{c} \begin{array}{ccccc} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{array} \\ \begin{pmatrix} p_{00} & p_{01} & p_{02} & p_{03} & p_{04} \\ p_{10} & p_{11} & p_{12} & p_{13} & p_{14} \\ p_{20} & p_{21} & p_{22} & p_{23} & p_{24} \\ p_{30} & p_{31} & p_{32} & p_{33} & p_{34} \\ p_{40} & p_{41} & p_{42} & p_{43} & p_{44} \end{pmatrix} \end{array}$$

Absorbing states have a 1 and 0's in their corresponding rows.

Placing the probabilities :

p_{10} : moving to the left with probability **0.5**

p_{12} : moving to the right with probability **0.5**

etc….

Movements on the matrix :

You've seen from the state transition diagram (**fig. 2.2)** that it is possible from each of the non-absorbing states **1,2** and **3** to reach the absorbing states **0** and **4**.You may see it on the matrix if you didn't draw a state transition diagram

1 → 0

2 → 1

3 → 2

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccccc} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Arrows show reaching state 0 from state 3: $\mathbf{3} \to \mathbf{2} \to 1 \to 0$

Example 2.9

In the following transition matrices, identify all absorbing chains and determine which chains are absorbing:

$$\textbf{1. } P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0.4 & 0.2 & 0.4 & 0 \\ 0 & 1 & 0 & 0 \\ 0.1 & 0 & 0.5 & 0.4 \\ 0.1 & 0 & 0.3 & 0.6 \end{pmatrix} \end{array}$$

To identify the absorbing states, look at the main diagonal of the matrix,where you see a 1 ,this means you have an absorbing state.

Here, since $\mathbf{p_{22} = 1}$,then we have an absorbing state at **2** .

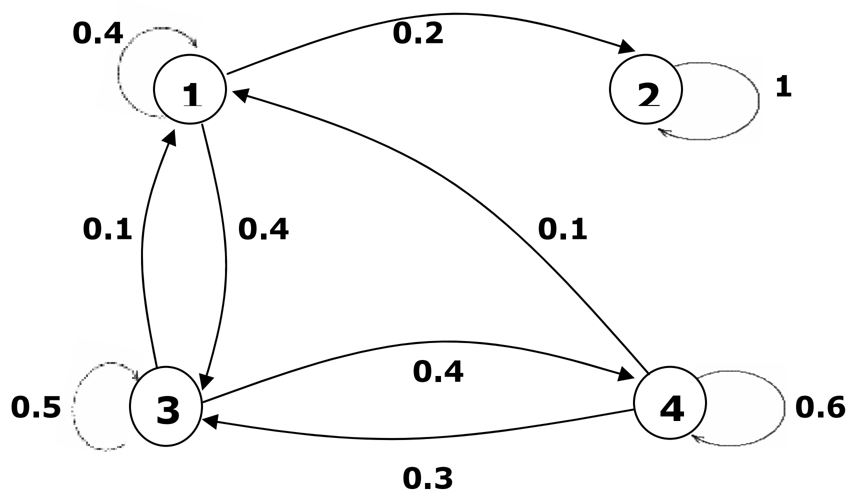
It is possible, from each of the non-absorbing states **1,3** and **4** to reach the absorbing state **2**: **1** $\to 2$ (directly) ; **3** $\to 1 \to 2$; **4** $\to 1 \to 2$ **Hence, the chain is an absorbing one.**

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0.4 & 0.2 & 0.4 & 0 \\ 0 & 1 & 0 & 0 \\ 0.1 & 0 & 0.5 & 0.4 \\ 0.1 & 0 & 0.3 & 0.6 \end{pmatrix} \end{array}$$

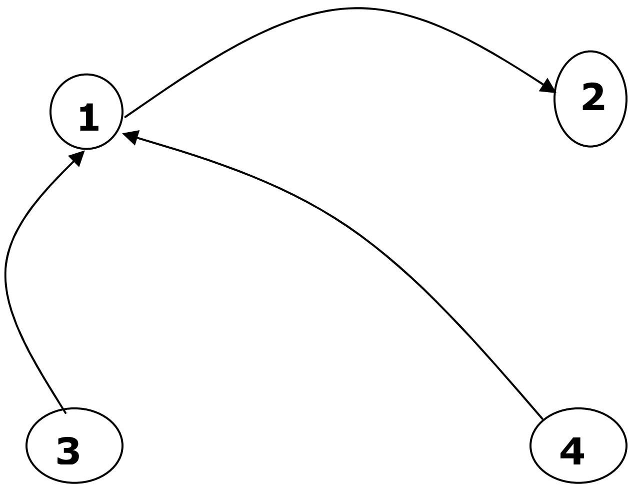
1 $\to 2$

3 $\to 1$ **4** $\to 1$

You may see it clearly when you draw a state transition diagram:



Path from the non-absorbing states 1 , 2 and 3 to the absorbing state 2 :



1 $\rightarrow \mathbf{2}$(directly); **3** $\rightarrow \mathbf{1} \rightarrow \mathbf{2}$;**4** $\rightarrow \mathbf{1} \rightarrow 2$

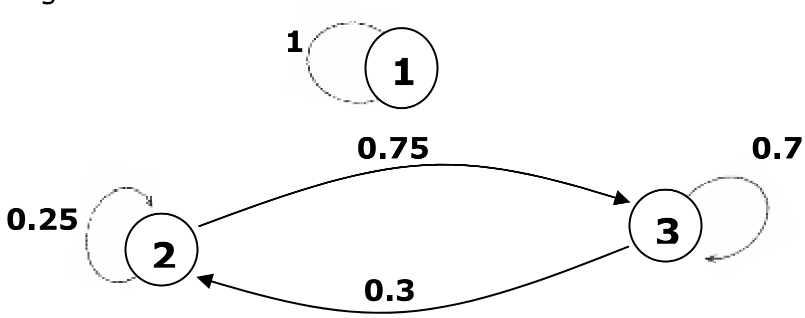
2. P =
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.25 & 0.75 \\ 0 & 0.3 & 0.7 \end{pmatrix} \end{array}$$

We have an absorbing state at **1** , since $\mathbf{p_{11} = 1}$

It is impossible to go from the non-absorbing state **2** to the absorbing state **1** ,Hence it is **not** an *absorbing chain*.

Note that also , it is impossible to go from the non-absorbing state **3** to the absorbing state **1.One case is enough to decide that the chain is not absorbing.**

The state transition diagram shows that it is **not** possible to reach the absorbing state 1 from either 2 or 3 .



Reflecting state

Definition: A **state** ***i*** is said to be a lower ***reflecting*** barrier if once you get to state **i** you can leave.

Example 2.10

In a Random walk ,the steps have probabilities **p,q,r** of taking the values **1,-1,0**:

Write down the transition matrices of the Markov Chain formed by this random walk in the following three cases:

a) Absorbing barriers at **-2** and **+2** :

$$\begin{array}{c} \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccccc} \mathbf{-2} & \mathbf{-1} & \mathbf{0} & \mathbf{1} & \mathbf{2} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ q & r & p & 0 & 0 \\ 0 & q & r & p & 0 \\ 0 & 0 & q & r & p \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

left move : **q** ; right move :**p** ,still: **r**

b) Reflective barriers at **-2** and **+2** such that if the system reaches a barrier at time **n** ,the system at time **n+1** is equally likely to occupy the states **1,0,-1**:

$$\begin{array}{c} \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccccc} \mathbf{-2} & \mathbf{-1} & \mathbf{0} & \mathbf{1} & \mathbf{2} \end{array} \\ \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ q & r & p & 0 & 0 \\ 0 & q & r & p & 0 \\ 0 & 0 & q & r & p \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix} \end{array}$$

Reflected to 1,-1,0 with equal Prob=1/3

c) Unrestricted process (No barriers):

$$\begin{array}{c} \\ . \\ -3 \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ . \end{array}\begin{array}{c} \begin{array}{cccccccc} \mathbf{..} & \mathbf{-3} & \mathbf{-2} & \mathbf{-1} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{..} \end{array} \\ \begin{pmatrix} .. & . & . & . & . & . & . & .. \\ .. & r & p & 0 & 0 & 0 & 0 & .. \\ .. & q & r & p & 0 & 0 & 0 & .. \\ .. & 0 & q & r & p & 0 & 0 & .. \\ .. & 0 & 0 & q & r & p & 0 & .. \\ .. & 0 & 0 & 0 & q & r & p & .. \\ .. & 0 & 0 & 0 & 0 & q & r & .. \\ .. & . & . & . & . & . & . & .. \end{pmatrix} \end{array}$$

Classification of states

- **Absorbing:** A state is ***absorbing*** if once the state is entered, it is impossible to leave it. State ***i*** is an absorbing state if and only if $P_{ii} = 1$.
- **Accessible :** A state ***j*** is said to be accessible from state ***i*** if one can get to ***j*** from ***i*** in some steps. **i.e.** $P_{ij} > 0$
- **Communicating :** If state **i** is accessible from state **j** and state **j** is accessible from state **i** **,**then states **i** and **j** are said to communicate.
- **Transient :** A state is said to be transient if upon entering this state,the process *may never return* to this state again.A state is transient if there exists a state ***j*** ($i \neq j$) that is accessible from ***i*** but not vice-versa.
- **Recurrent :** A state is said to be recurrent if upon entering this state, the process definitely will return this state again.
- **Periodic :** A state is said to be periodic if upon entering this state ,the process definitely return to this state in fixed number of steps.
- **Irreducible Markov Chain:** If all states communicate then the Markov chain can not be simplified and is said to be irreducible.

Calculation of Probabilities (Absorbing Chain)

Question: Suppose we start in a non-absorbing state within an absorbing system. We know that eventually it will be our fate to wind up in one of the absorbing states.

How long can we expect to be around before absorption?

(in the gamble's ruin example : after what time one of the players will go bankrupt?)

How many times can we expect to visit each of the other non-absorbing states before absorption happens?

The answer is amazingly ,even magically easy to compute.

Let's look again at the drunkard's walk example:

In an absorbing system ,**Re-Number** the states so that the **non-absorbing** state **comes first**.

$$\mathbf{P} = \begin{matrix} & 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0.5 & 0 & 0.5 & 0 & 0 \\ 2 & 0 & 0.5 & 0 & 0.5 & 0 \\ 3 & 0 & 0 & 0.5 & 0 & 0.5 \\ 4 & 0 & 0 & 0 & 0 & 1 \end{matrix} \rightarrow \begin{matrix} & 1 & 2 & 3 & 0 & 4 \\ 1 & 0 & 0.5 & 0 & 0.5 & 0 \\ 2 & 0.5 & 0 & 0.5 & 0 & 0 \\ 3 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 1 \end{matrix}$$

$$\mathbf{P} = \begin{matrix} & 1 & 2 & 3 & 0 & 4 \\ 1 & 0 & 0.5 & 0 & 0.5 & 0 \\ 2 & 0.5 & 0 & 0.5 & 0 & 0 \\ 3 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 1 \end{matrix}$$

Notice in every case, the transition matrix **P** has a square identity matrix in the bottom right corner (because of the absorbing states).
Notice also that there is a block of zeros in the bottom left next to the identity matrix (reflecting the fact that you can never leave an absorbing state)
Symbolically, we can break **P** down into the following blocks:

$$\mathbf{P} = \begin{pmatrix} \boldsymbol{S} & \boldsymbol{T} \\ 0 & \boldsymbol{I} \end{pmatrix}$$

where :

I : is an **m x m** identity matrix (m is the number of absorbing states)

S : is a square **(n – m) x (n – m)** matrix where **n** is the total number of states.
m : is the number of absorbing states
Therefore , **n – m** is the number of non-absorbing states.

The matrix S gives the transition probabilities for movement among the non-absorbing states.

Remark: the size of I is the number of the absorbing states.

If we have a single absorbing state then I = (1)

see e.g. 2.7 below

The Fundamental matrix

For an absorbing Markov system ,the Fundamental matrix is the matrix : **Q = (I – S)$^{-1}$**

Where **I** is the identity matrix with the same dimensions as **S**.

If we are at a *non-absorbing* state i at time 0 ,the number of time steps we can expect, on average, to land at *non-absorbing* state j before absorption is given by the ijth entry in the fundamental matrix.

Remark : If i = j ,then this includes the start time.

Example 2.11

Consider the transition matrix **P =** $\begin{matrix}1\\2\\3\end{matrix}\left(\begin{array}{cc|c}0.9 & 0.1 & 0\\ 0.1 & 0.7 & 0.2\\ \hline 0 & 0 & 1\end{array}\right)$

We have here, a single absorbing state at **3** since $\mathbf{P_{33} = 1}$

then **S =** $\begin{pmatrix}0.9 & 0.1\\ 0.1 & 0.7\end{pmatrix}$ **and I – S =** $\begin{pmatrix}0.1 & -0.1\\ -0.1 & 0.3\end{pmatrix}$

det (I – S) = (0.1)(0.3) – (- 0.1)(- 0.1) = 0.02

Q = (I – S)$^{-1}$ = $\dfrac{1}{0.02}\begin{pmatrix}0.3 & 0.1\\ 0.1 & 0.1\end{pmatrix}$ **=** $\begin{pmatrix}15 & 5\\ 5 & 5\end{pmatrix}$

$\mathbf{Q_{11}}$ **= 15 :** Starting from **state 1 ,**we will land in **state 1** an average **15** time steps *including starting time* **(see the last remark)** before **absorption.**

Q_{12} = 5 : Starting in **state 1 ,**we will land in **state 2** an average of **5** time steps before absorption.

Q_{21} = 5 : Starting in **state 2 ,**we will land in **state 1** an average of **5** time steps before absorption.

Q_{22} = 5 : Starting from **state 2 ,**we will land in **state 1** an average **5** time steps *including starting time* before **absorption.**

Example 2.12

In the Drunkard's Walk example, the transition matrix:

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 0 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 0 & 4 \end{array} \\ \left(\begin{array}{ccc|cc} 0 & 0.5 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \end{array}$$

$$\mathbf{S} = \begin{pmatrix} 0 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0 \end{pmatrix}; \quad \mathbf{I - S} = \begin{pmatrix} 1 & -0.5 & 0 \\ -0.5 & 1 & -0.5 \\ 0 & -0.5 & 1 \end{pmatrix};$$

$$\mathbf{Q = (I - S)^{-1}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1.5 & 1 & 0.5 \\ 1 & 2 & 1 \\ 0.5 & 1 & 1.5 \end{pmatrix} \end{array}$$

If we start in state **2** then the expected number of times to be in state **3** before being absorbed is **Q_{23} = 2 .**

Time to absorption

Question: Given that the chain starts in state **i** ,what is the expected number of steps before the chain is absorbed?

Answer: Add the entries in the rows of the fundamental matrix.

(Based on a theorem which we will accept its result without proof).

In the drunkard's walk example, we found:

$$\mathbf{Q} = (\mathbf{I} - \mathbf{S})^{-1} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{pmatrix} 1.5 & 1 & 0.5 \\ 1 & 2 & 1 \\ 0.5 & 1 & 1.5 \end{pmatrix} \end{array}$$

starting in state **1** , the expected number of steps to absorption

=1.5+1+0.5 = 3 i.e. if we start at state **1 ,**absorption will occur at **t=3**

Similarly, times to absorption starting in **2** , **3** are **4 , 3** respectively.

Absorption Probabilities

Question: What is the probability that an absorbing chain will be absorbed in an absorbing state **j** if it starts in a transient state **i** ?

Answer: The entries of the matrix **B** = **Q.T** ,where **Q** is the fundamental matrix and **T** is the matrix that appears in the form :

$$\mathbf{P} = \begin{pmatrix} S & T \\ 0 & I \end{pmatrix};$$

(Based on a theorem which we will accept its result without proof).

In the drunkard's walk example, we found:

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 0 \\ 4 \end{array}\!\!\begin{array}{c} \begin{matrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{0} & \mathbf{4} \end{matrix} \\ \left(\begin{array}{ccc|cc} 0 & 0.5 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \end{array}; \quad \mathbf{Q} = \begin{pmatrix} 1.5 & 1 & 0.5 \\ 1 & 2 & 1 \\ 0.5 & 1 & 1.5 \end{pmatrix} \text{(calculated above)}$$

$$\text{Here, } \mathbf{T} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 0 & 4 \end{array} \\ \begin{pmatrix} 0.5 & 0 \\ 0 & 0 \\ 0.5 & 0 \end{pmatrix} \end{array}$$

$$\mathbf{B = Q.T} = \begin{pmatrix} 1.5 & 1 & 0.5 \\ 1 & 2 & 1 \\ 0.5 & 1 & 1.5 \end{pmatrix} \begin{pmatrix} 0.5 & 0 \\ 0 & 0 \\ 0.5 & 0 \end{pmatrix} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 0 & 4 \end{array} \\ \begin{pmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \\ 0.25 & 0.75 \end{pmatrix} \end{array}$$

Therefore ,

Starting in **1** , there is a probability **0.75** of absorption in state **0**
and a probability of **0.25** of absorption in state **4**

Starting in **2** , there is a probability **0.5** of absorption in state **0**
and a probability of **0.5** of absorption in state **4**

Starting in **3** , there is a probability **0.25** of absorption in state **0**
and a probability of **0.75** of absorption in state **4**

Average Time between visits

Question : Given we are in state **i** and leave, what is the average time that elapses before we return again to state **i** ?

Answer: The average amount of time elapsed between visits to state ***i*** (called ***mean recurrence time***) is given by the reciprocal of the ***i* th** component of the fixed probability vector.

(Based on a theorem which we will accept its result without proof).

More Examples on Markov Chains

Example: Gambler's Ruin

Gamblers: A, B have a total of N dollars

Game: Toss Coin

If $H \Rightarrow A$ receives \$1 from B

$T \Rightarrow B$ receives \$1 from A

$$P(H) = p, \quad P(T) = q = 1 - p$$

$X_n =$ Amount of money A has after n plays

$$P\{X_{n+1} = X_n + 1 | X_n\} = p$$

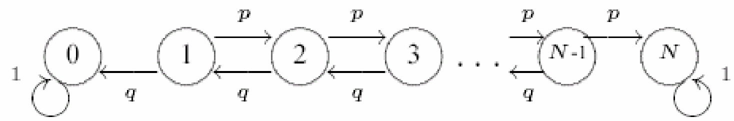
$$P\{X_{n+1} = X_n - 1 | X_n\} = q$$

.....Game ends if $X_n = 0$ or $X_n = N$

State space $= \{0, 1, 2, \ldots, N\}$

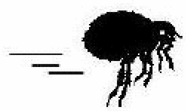
$$X_n \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N \end{array} \overset{X_{n+1}}{\begin{array}{cccccccc} \underline{0} & \underline{1} & \underline{2} & \underline{3} & \cdots & \underline{N-2} & \underline{N-1} & \underline{N} \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ q & 0 & p & 0 & \cdots & 0 & 0 & 0 \\ 0 & q & 0 & p & \cdots & 0 & 0 & 0 \\ & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & q & 0 & p \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{array}}$$

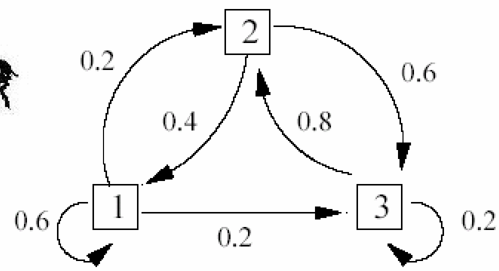
Transition Diagram for Gambler's Ruin



Example2

Purpose-flea zooms around the vertices of the transition diagram opposite. Let X_t be Purpose-flea's state at time t $(t = 0, 1, \ldots)$.





Find the transition matrix, P.

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{pmatrix}$$

Example 3

Simple random walk with absorbing barriers at **- 2** and **+ 2** .

Here the state space is **{-2 ,-1,0,1,2}.**

State **-2** , **+ 2** being absorbing barriers mean that once the object reaches **-2** or **+2** it remains there.While the object is at one of the intermediate states ,the probabilities of moving one step to the right/left/not moving are **p,q,r** with **p+q+r = 1**.The transition matrix :

$$\mathbf{P} = \begin{array}{c} \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccccc} -2 & -1 & 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ q & r & p & 0 & 0 \\ 0 & q & r & p & 0 \\ 0 & 0 & q & r & p \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Example 4 (Ehrenfest model)

The following is a special case of a model called, the Ehrenfest model, that has been used to explain diffusion of gases.

There are **N** particles in a container which has a permeable partition. At each step one of the particles is chosen at random passes through the partition. Here we record the state as the number of particles to one side of the partition, so **S = {0 ,1 ,2,3,…,N}.** When the stat is **i** ,there is a probability **i/N** that the next state is **i – 1** and a probability **1-(i/N)** that the next state is **i + 1**

Consider the model of state space **S = {0 ,1 ,2,3,4}**

Since the particles have to be moved, P(not moving) = 0

i.e. $p_{00} = p_{11} = p_{22} = p_{33} = p_{44} = 0$

Remember: A transition matrix is a one step movement.

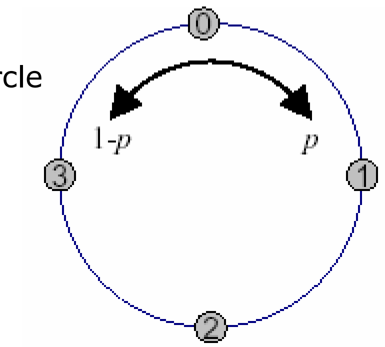
So $p_{02} = p_{03} = p_{04} = 0 \Rightarrow p_{01} = 1$ similarly $p_{43} = 1$

The transition matrix:

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Example 5

A particle moves on a circle at points which have been marked 0,1,2 and 3 in a clockwise order. At each step, the particle has a probability *p* of moving clockwise and 1- *p* of moving counterclockwise:



Let X_n denote the location of the particle on the circle after the *n*th step. The process $X_n;\ n = 0;\ 1;\ 2;\ ..$ is a Markov chain.

The transition matrix :

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{matrix} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \end{matrix} \\ \begin{pmatrix} 0 & p & 0 & 1-p \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ p & 0 & 1-p & 0 \end{pmatrix} \end{array}$$

EXERCISES

Hints & Answers on page 42

1. Consider a community with three grocery stores. Within this community(we assume the population is fixed) there always exists shift of customers from one grocery to another. A study was made on January 1 ,and it was found that **¼** of the population shopped at store **I** , **1/3** at store **II** and **5/12** at store **III**.Each month store **I** retains **90%** of its customers and loses **10%** of them to store **II**. Store **II** retains **5%** of its customers and loses **85%** of them to store **I** and **10%** of them to store **3**.Store III retains 40% of its customers and loses **50%** of them to store **I** and **10%** of them to store **II:**

a) Write the transition matrix of the above process.
b) What proportion of customers will each store retain by February 1?
c) Assuming the same pattern continues, what will be the long-run distribution of customers among the three stores?

2. A salesman lives in town A and is responsible for towns A,B and C. Each week he is required to visit a different town. When he is in his home town, it doesn't make a difference which town he visits next so he flips a coin and if it is Heads he goes to B,and if tails he goes to C. After spending a week away from his home, so when he is either in B or C ,he flips two coins. If two heads occur ,then he goes to the other town, otherwise he goes home:

a) Write the transition matrix for the above Markov chain.
b) Draw a suitably annotated state transition diagram for the above chain.

3. **i.** Give a definition of the term simple random walk.
Explain what is meant by an absorbing barrier.
ii. Give a definition of a Markov Chain and a transition matrix.

iii. State the Chapman-Kolmogorov equations.

4. When a motorist's insurance policy comes up for renewal, the probability that he is satisfied with the insurance company and renews the policy is 50% if he is insured with company A, or 75% if the policy is with companies B or C. If he is not satisfied he looks at the policies of the other companies; the conditional probabilities of taking out a policy with A, B or C are in the ratio 3 : 2 : 1. (For example, if he is dissatisfied with A he will not choose A next, so the conditional probability of choosing C would be 1 / 2+1 .) Let X_n denote the company with which the motorist is insured in year n.

a) Write the transition matrix for the above Markov chain.
b) Find the stationary distribution of X and state, with reasons, whether it is a limiting distribution.
c) Find the probability that, if the motorist is insured with Company B this year, he will be insured with company C in three years' time.

5. For the following transition matrices, identify all the absorbing states and determine which of the Markov chains are absorbing:

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0.4 & 0.3 & 0.4 \\ 0.2 & 0.7 & 0.1 \end{pmatrix} \; ; \; P_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.25 & 0 & 0.75 & 0 \end{pmatrix}$$

$$P_3 = \begin{pmatrix} 0.3 & 0.7 & 0 \\ 0.7 & 0.3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; ; \; P_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1/8 & 5/8 & 2/8 \\ 0 & 0 & 1 \end{pmatrix}$$

6. Consider the absorbing Markov chain with the following transition matrix between states A,B,C and D :

$$\begin{pmatrix} 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

a) Compute the fundamental matrix **Q** .

b) Determine the number of times stating in state B ,the system will re visit state B prior to absorption.

c) Suppose the system is in state C at t = 0 ,what is the expected time before the chain is absorbed?

7. Gambler's Ruin : Player 1 has \$ 3 and Player 2 has \$ 2.They flip a coin, if it is Head Player 2 wins \$1 ,if it is a tail Player 1 wins \$1 :

a) Write down the transition matrix of the chain.
of course : two absorbing states at 0 and 5 ; $p = q = ½$

b) Starting with \$ 3 ,what is the expected number of times the player will have \$1 before the termination of the game(absorption)?
(Note: This is the same as the number of times he will hold between \$1 and \$4 inclusive before the game ends).

c) Starting with \$ 3 ,What is the expected number of games before absorption (game length)?

d) What is the probability that an absorbing state is reached(one of the players is ruined)?

8. Which of the following Markov chains is regular?

$$\mathbf{P_1} = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix} \qquad \mathbf{P_2} = \begin{pmatrix} 1 & 0 \\ 3/4 & 1/4 \end{pmatrix}$$

$$\mathbf{P_3} = \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1 & 0 \end{pmatrix} \qquad \mathbf{P_4} = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 3/4 & 0 & 1/4 \\ 0 & 1 & 0 \end{pmatrix}$$

For $\mathbf{P_4}$ calculate the limiting probability vector π

9. A car insurance company operates a no-claims discount scheme with three levels: No discount, Partial discount and Full discount. A policyholder who goes a whole year without an accident advances to the next discount level, unless already on Full discount; a policyholder who has one accident in a year is moved to the next lower discount level, unless already at No discount; a policyholder who has more than one accident in a year is moved to No discount.

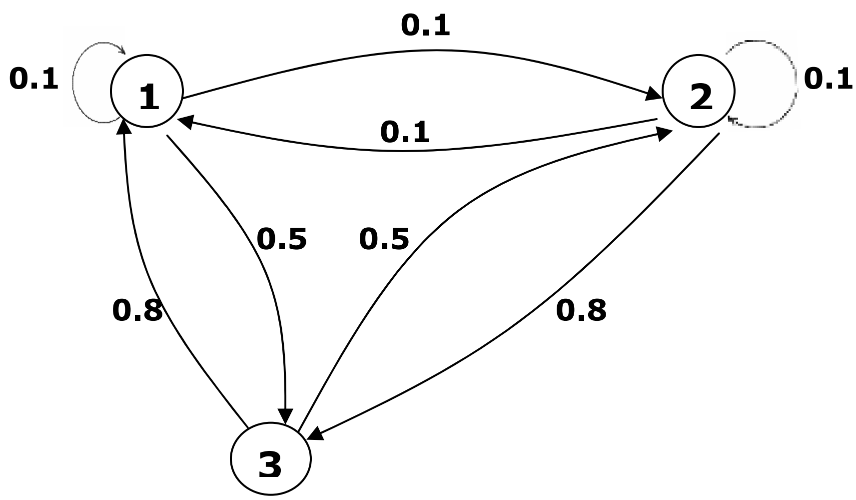
a) If Xn is the policyholder's discount level in year n, formulate X as a discrete-time Markov chain and write down its transition matrix in the case of a policyholder who, in any given year, has **0** accidents with probability θ, **1** accident with probability **1 -** θ, more than one accident with probability **(1-**θ**)**^{2}. In the case θ = **0.9** evaluate the long-run proportion of time the customer spends in each of the three possible discount states.

b) If a driver takes out a new policy with the company in year **0** with no discount, find the probabilities that the driver is in each of the possible discount states after three years, assuming

i. $\theta = 0.9$ **ii.** $\theta = 0.4$.

In which case would you say that convergence to equilibrium is faster?

10. Consider the Markov system represented by the following diagram:



a) Write down the transition matrix for this system and compute the long-term transition matrix.

b) What fraction of time does the system spend in state **2** ?

c) For a system starting in state **1** , which is more likely : that the system is in state **2** after two steps or that the system is in state **2** after large number of steps ? How likely is that ?

d) Given the initial distribution vector **[5 4 9]** calculate its equilibrium distribution.

e) Consider the two – state Markov system obtained by combining state **1** and state **2** in a single state **A** , and of state **2** as the single state **B** What is the transition matrix for this system? Without any further computation, write the long – term transition matrix.

Hints and Answers to Exercises:

1. a) $\begin{array}{c} \\ I \\ II \\ III \end{array} \begin{array}{c} \begin{array}{ccc} \textbf{I} & \textbf{II} & \textbf{III} \end{array} \\ \begin{pmatrix} 0.9 & 0.1 & 0 \\ 0.85 & 0.05 & 0.1 \\ 0.5 & 0.1 & 0.4 \end{pmatrix} \end{array}$ **; b) [0.7167   0.8333   0.2]**

c) Store I : 88.9% ; II : 9.5 % ; III : 1.6%

2. $\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} \textbf{A} & \textbf{B} & \textbf{C} \end{array} \\ \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.75 & 0 & 0.25 \\ 0.75 & 0.25 & 0 \end{pmatrix} \end{array}$

3. i. A simple random walk is a discrete-time Markov process $\mathbf{X_n}$ with transition probabilities $\mathbf{p_{i,i+1} = p,\ p_{i,i-1} = q = 1 - p}$.
An absorbing barrier is a state i such that $p_{ii} = 1$.

ii. A Markov chain is a discrete-time, discrete state space process X such that $P(X_{t+1}=s \mid X_t = s_t ; X_{t-1} = s_{t-1} ;\dots; X_1 = s_1; X_0 = s_0) = P(X_{t+1} = s \mid X_t = s_t)$
It follows that X is determined by pij, $= P(X_{n+1} = j \mid X_n = i)$, which can be assembled into a matrix Pn, the transition matrix.

iii. The Chapman-Kolmogorov equations state that $P^{m+n} = P^nP^m$, where P^n is the n-step transition matrix. In other words, that

$$P(X_{m+n} = j \mid X_0 = i) = \sum_k P(X_m = k \mid X_0 = i)P(X_{m+n} = j \mid X_m = k).$$

It is clear that $P^1= P$, the transition matrix;
from the Chapman- Kolmogorov equations we have $P^{n+1} = P^1P^n$

4. a) P = $\begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 3/16 & 3/4 & 1/16 \\ 3/20 & 1/10 & 3/4 \end{pmatrix}$ Regular so limiting

b.) (9/35 ,16/35 ,10/35)

c.) Calculate P^3 ,Ans. = 0.174

5. P_1 : Absorbing ; P_2 : Not absorbing ; P_3 :Not Absorbing ; P_4 : Absorbing ;

6. a) Q = $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}$; b) Twice ; c) t = 4

7. a) $\begin{matrix} & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 2 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 3 & 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 4 & 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 5 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}$ **;**

b)0.8 ; c) 6 ; d) 0.6 player1 wins

8. P_1 , P_4 are regular ; P_2 , P_3 are not. π (1/3 , 1/3 ,1/3)

9. a) **P =** $\begin{pmatrix} 1-\theta & \theta & 0 \\ 1-\theta & 0 & \theta \\ (1-\theta)^2 & \theta(1-\theta) & 0 \end{pmatrix}$; (0.021 , 0.098, 0.881)

b) We need the top row of $\mathbf{P^3}$

For θ = 0.9 this is (.0271, .1629, .81) and for θ = 0.4 it is (.5616, .2784, .16). The convergence seems slightly faster in the θ = 0.4 case.

10. a) **P =** $\begin{matrix} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ 1 & 0.1 & 0.1 & 0.8 \\ 2 & 0.1 & 0.1 & 0.8 \\ 3 & 0.5 & 0.5 & 0 \end{matrix}$; P^∞ **=** $\begin{pmatrix} 5/18 & 5/18 & 4/9 \\ 5/18 & 5/18 & 4/9 \\ 5/18 & 5/18 & 4/9 \end{pmatrix}$

b) 5/18 ;

c) The system is in state **2** after 2 steps with probability 21/50

d) [5 5 8] ;

e) P = $\begin{pmatrix} 0.2 & 0.8 \\ 1 & 0 \end{pmatrix}$ **;** $P^\infty = \begin{pmatrix} 5/9 & 4/9 \\ 5/9 & 4/9 \end{pmatrix}$

Chapter 3: Applications on Markov Chains

What's in this Chapter?

- **Gambler's Ruin ;**
- **The Poisson Process;**
- **Queuing Theory;**
- **Birth-Death Problems**.

Revision : Expectation

The **mean**, **expected value**, or **expectation** of a random variable **X** is written as **E(X)** or μ_X. If we observe **N** random values of **X**, then the mean of the **N** values will be approximately equal to **E(X)** for large **N**. The expectation.

Definition: Let **X** be a discrete random variable. The expected value of **X** is

E(X) = $\sum_x xP(X = x)$

Recall that :**X** is a discrete random variable that can take only values $x_1, x_2, \ldots, x_n$,with probabilities $p_1, p_2, \ldots, p_n$ associated with these values **if** $p_1 + p_2 + \ldots + p_n = 1$ .The expectation or the expected value of X:

E(X) = $\sum x_i p_i = x_1p_1 + x_2p_2 + \ldots + x_np_n$

Expectation as a Probability

Let A be any event. We can write P(A) as an expectation, as follows. Define the indicator function: I_A = **1** if event **A** occurs, **0** otherwise:

Then I_A is a random variable, and

$$E(I_A) = \sum_{r=0}^{1} rP(I_A = r) = 0 \times P(I_A = 0) + 1 \times P(I_A = 1) = P(I_A = 1) = P(A)$$

Thus $P(A) = E(I_A)$ for any event **A** .

3.1 The Gambler's Ruin

A phenomenon of probability called "gambler's ruin" is one of the things that keeps me from gambling. Essentially, gambler's ruin says that, if you play long enough, you will go bankrupt, and have to quit (as the house will not extend you credit). "Long enough" may be a very long time. It mainly depends on how much money you start with, how much you bet, and the odds of the game. Even with better than even odds, you will eventually go bankrupt, and have to quit. But, this may take very long, indeed.

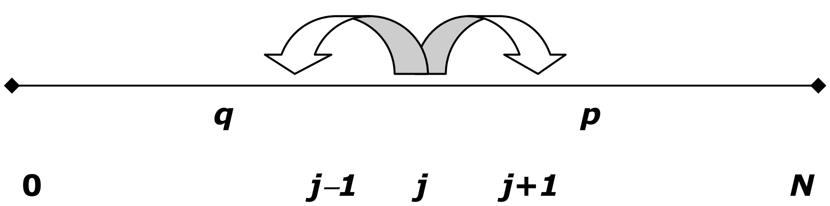
Gambler **A** has **$ j**
Gambler **B** has **$ N –j**
Total = **$ N** between them
Bet **$1** at a time

Gambler **A** increases capital to **$ j+1** with probability ***p***
Gambler **A** decreases capital to **$ k–1** with probability ***q=1–p***

Successive bets are independent

Repeat procedure until one player is out of money so that the gambler has either **$ 0 or $ N**

Want to know: P(gambler ruins)=P(…ends up with **$ 0**).



Consider the Ruin of A : Let θ_j be the probability **A** is ruined before **B**

Reaching 0 before N , with boundary conditions θ_0 **= 1 and** θ_N **= 0**

$\theta_j = p\theta_{j+1} + q\theta_{j-1} \Rightarrow p\theta_{j+1} - \theta_j + q\theta_{j-1} = 0$ **,this is a second order difference equation.**

The Auxiliary equation : $pr^2 - r + q = 0 \Rightarrow (pr - q)(r - 1) = 0$

Hence , two real distinct roots exist : $r_1 = 1$, $r_2 = q/p$

The general solution is : $\theta_j = C_1 r_1^j + C_2 r_2^j$; C_1 and C_2 are constants.

$\theta_j = C_1(1)^j + C_2\left(\frac{q}{p}\right)^j = C_1 + C_2\left(\frac{q}{p}\right)^j$; using the boundary conditions :

$\theta_0 = 1 \Rightarrow 1 = C_1 + C_2$ and $\theta_N = 0 \Rightarrow 0 = C_1 + C_2\left(\frac{q}{p}\right)^N$

Solving simultaneously: $\mathbf{C_1} = \dfrac{-\left(\frac{q}{p}\right)^N}{1-\left(\frac{q}{p}\right)^N}$; $\mathbf{C_2} = \dfrac{1}{1-\left(\frac{q}{p}\right)^N}$

Substituting **C_1 and C_2** in the solution **:**

$$\theta_j = C_1 + C_2\left(\frac{q}{p}\right)^j = \frac{-\left(\frac{q}{p}\right)^N}{1-\left(\frac{q}{p}\right)^N} + \frac{\left(\frac{q}{p}\right)^j}{1-\left(\frac{q}{p}\right)^N} = \frac{\left(\frac{q}{p}\right)^j - \left(\frac{q}{p}\right)^N}{1-\left(\frac{q}{p}\right)^N} \quad \textbf{if } p \neq q$$

If p = q =1/2 ,then q/p =1 and both numerator & denominator= 0

Let x = q/p , then x $\to$ 1 ; $\theta_j = \dfrac{x^j - x^N}{1 - x^N}$, Now Using **L'Hopital** rule :

$$\lim_{x\to 1}\frac{x^j - x^N}{1-x^N} = \frac{jx^{j-1} - Nx^{N-1}}{Nx^{N-1}} = \frac{j(1) - N(1)}{N(1)} = \frac{j-N}{N}$$

<u>Another Solution for the Case p = q</u>

When **p = q**, then the roots are **$r_1 = 1$, $r_2 = q/p = 1$**

Hence ,we have two **equal** real roots the solution of the difference equation

is $\theta_j = (C_1 + C_2 j)r^j = (C_1 + C_2 j)(1)^j = C_1 + C_2 j$ **,** using the boundary

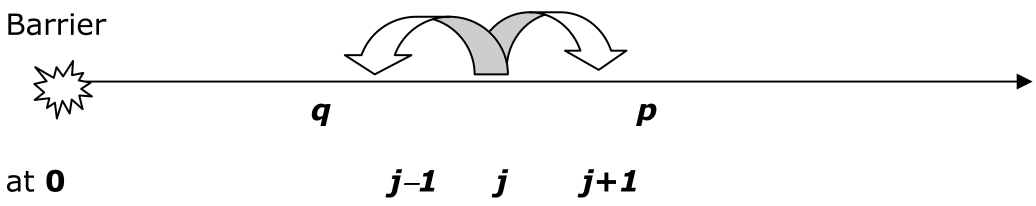
conditions : $\theta_0 = 1 \Rightarrow$ **1 = C_1 + C_2 (0)** $\Rightarrow$ **C_1 = 1**

$\theta_N = 0 \Rightarrow$ **0 = C_1 + C_2(N)** $\Rightarrow$ **C_2 = -1/N**

Substituting **C_1** and **C_2** in the solution : θ_j **= 1- ($\frac{1}{N}$)j =** $\dfrac{N-j}{N}$

Case of One Absorbing barrier

Suppose the amount to play with is not limited (no upper barrier), in this case we have one absorbing barrier at **0** only and **N** $\to \infty$**:**



For the random walk with absorbing barrier at **0** but no upper barrier

P(Absorbed at 0 | start at j) = $\begin{cases} \left(\dfrac{q}{p}\right)^j & \text{if } q < p \\ 1 & \text{if } q \geq p \end{cases}$

<u>Proof</u>: for p $\neq$ **q**

Case 1: q < p then **Remember** $\dfrac{q}{p}$ **< 1 ,then** $\left(\dfrac{q}{p}\right)^{\infty} \to 0$

$$\theta_j = \frac{\left(\frac{q}{p}\right)^j - \left(\frac{q}{p}\right)^N}{1-\left(\frac{q}{p}\right)^N} \xrightarrow{N\to\infty} \frac{\left(\frac{q}{p}\right)^j - 0}{1-0} = \left(\frac{q}{p}\right)^j;$$

Case 2: q > p then **Remember** $\dfrac{q}{p}$ **> 1 ,then** $\left(\dfrac{q}{p}\right)^{\infty} \to \infty$

θ_j **=**

$$\frac{\left(\frac{q}{p}\right)^j - \left(\frac{q}{p}\right)^N}{1-\left(\frac{q}{p}\right)^N} \xrightarrow{N\to\infty} \frac{\left(\frac{q}{p}\right)^j}{1-\left(\frac{q}{p}\right)^N} - \frac{\left(\frac{q}{p}\right)^N}{1-\left(\frac{q}{p}\right)^N} = 0 - \frac{\left(\frac{q}{p}\right)^N}{\left(\frac{q}{p}\right)^N\left(\left(\frac{p}{q}\right)^N - 1\right)} = 1$$

Case 3 : p=q=½ : $\theta_j = \dfrac{N-j}{N} = 1 - \dfrac{j}{N} \xrightarrow{N\to\infty}$ **1**

Summary A's Ruin

1. Two absorbing barriers: $\theta_j = \dfrac{\left(\dfrac{q}{p}\right)^j - \left(\dfrac{q}{p}\right)^N}{1-\left(\dfrac{q}{p}\right)^N}$ **if p $\neq$ q**

$= \dfrac{N-j}{N}$ **if p = q**

2. Single Absorbing barrier at 0 : $\theta_j = \begin{cases} \left(\dfrac{q}{p}\right)^j & if\ q<p \\ 1 & if\ q \geq p \end{cases}$

Consider the Ruin of B:

Reaching N before 0 ,with boundary conditions θ_0 **= 0 and** θ_N **= 1**

θ_j = **p**θ_{j+1} + **q**$\theta_{j-1} \Rightarrow$ **p**θ_{j+1} **-** θ_j **+ q**θ_{j-1} **= 0 ,this is a second order difference equation.**

The Auxiliary equation : $pr^2 - r + q = 0 \Rightarrow (pr - q)(r - 1) = 0$

Hence , two real distinct roots exist : $r_1 = 1$, $r_2 = q/p$

The general solution is : $\theta_j = C_1 r_1^j + C_2 r_2^j$ **;** C_1 and C_2 are constants.

$\theta_j = C_1(1)^j + C_2\left(\dfrac{q}{p}\right)^j = C_1 + C_2\left(\dfrac{q}{p}\right)^j$ **;** using the boundary conditions :

$\theta_0 = 0 \Rightarrow 0 = C_1 + C_2 \Rightarrow C_2 = -C_1$

$\theta_N = 1 \Rightarrow 1 = C_1 + C_2\left(\dfrac{q}{p}\right)^N = C_1 - C_1\left(\dfrac{q}{p}\right)^N$ **replacing C_2 by $-C_1$**

1 = $C_1\left[1-\left(\frac{q}{p}\right)^N\right] \Rightarrow C_1 = \frac{1}{1-\left(\frac{q}{p}\right)^N}$ **;** $C_2 = -C_1 = \frac{-1}{1-\left(\frac{q}{p}\right)^N}$

Thus θ_j **=** $\frac{1}{1-\left(\frac{q}{p}\right)^N}$ **-** $\frac{-1}{1-\left(\frac{q}{p}\right)^N}\left(\frac{q}{p}\right)^j$ **=** $\frac{1}{1-\left(\frac{q}{p}\right)^N}\left(1-\left(\frac{q}{p}\right)^j\right)$

$$\theta_j = \frac{1-\left(\frac{q}{p}\right)^j}{1-\left(\frac{q}{p}\right)^N} \quad \textbf{if} \quad p \neq q$$

If p = q =1/2 ,then q/p =1 and both numerator & denominator= 0

Let x = q/p , then x $\rightarrow$ **1**

θ_j **=** $\frac{1-x^j}{1-x^N}$, Now Using **L'Hopital** rule : $\lim_{x\to 1}\frac{1-x^j}{1-x^N} = \frac{jx^{j-1}}{Nx^{N-1}} = \frac{j(1)}{N(1)} = \frac{j}{N}$

__Another Solution for the Case p = q__

When **p = q**, then the roots are **r_1 = 1 , r_2 = q/p = 1**

Hence ,we have two **equal** real roots the solution of the difference equation

is θ_j **=** $(C_1 + C_2 j)r^j = (C_1 + C_2 j)(1)^j = C_1 + C_2 j$ **,** using the boundary

conditions : $\theta_0 = 0 \Rightarrow$ **0 = C_1**

$\theta_N = 1 \Rightarrow$ **1 = C_1 + C_2(N)** $\Rightarrow$ **1 = 0 + C_2(N)** $\Rightarrow$ **C_2 = 1/N**

Substituting **C_1** and **C_2** in the solution : θ_j **= 0+ ($\frac{1}{N}$)j =** $\frac{j}{N}$

Case of Single Absorbing barrier at 0

With a similar argument of **A's** case: $\theta_j = \begin{cases} \left(\dfrac{q}{p}\right)^j & if\ q < p \\ 1 \quad if\ q \geq p \end{cases}$

Summary : B's Ruin

1. Two Absorbing barriers : $\theta_j = \dfrac{1-\left(\dfrac{q}{p}\right)^j}{1-\left(\dfrac{q}{p}\right)^N} \quad if \quad p \neq q$

$= \dfrac{j}{N}$ if p = q

2. Single Absorbing barrier: $\theta_j = \begin{cases} \left(\dfrac{q}{p}\right)^j & if\ q < p \\ 1 \quad if\ q \geq p \end{cases}$

Example 3.1

Assume each player stands **$1** to win each play.

Initially player **A** has **$4** and player **B** has **$3**

The game continues until one of the players is ruined.

Answer the following questions:

a) What is the probability that **B** will eventually be ruined knowing that the odds are fair such that each player wins with a probability of **0.5** ?

b) What is the probability that **A** will eventually be ruined knowing that the odds are biased such that each play **A** wins with a probability of **0.6** and **B** wins with a probability of **0.4** ?

a) **j = 4 , N = 7 , p = 0.6 and q = 0.4 (p = q)**

P(B is ruined) = $\dfrac{j}{N} = \dfrac{4}{7}$

b) **j = 4 , N = 7 , p = 0.6 and q = 0.4 (p $\neq$ q)**

$$\textbf{P(A is ruined)} = \frac{\left(\frac{q}{p}\right)^j - \left(\frac{q}{p}\right)^N}{1-\left(\frac{q}{p}\right)^N} = \frac{\left(\frac{0.6}{0.4}\right)^4 - \left(\frac{0.6}{0.4}\right)^7}{1-\left(\frac{0.6}{0.4}\right)^7} = \textbf{0.148}$$

Example 3.2

A man repeatedly bets **$1** each day .If he wins, he wins **$1**.He stops playing when he goes broke or when he accumulates **$3**.His probability of winning is **0.4** .What is the probability of eventually accumulating **$3** if he starts with :

a) $ 1 ; b) $ 2

a) j = 1 , N = 3 , p = 0.4 , q = 0.6 ; reaching N before 0

(B's case with p $\neq$ q)

Probability (reaching 3 before 0| starting at 1)

$$= \frac{1-\left(\frac{q}{p}\right)^j}{1-\left(\frac{q}{p}\right)^N} = \frac{1-(3/2)^1}{1-(3/2)^3} = \frac{4}{19} \approx 0.2105$$

a) j = 2 , N = 3 , p = 0.4 , q = 0.6 ; reaching N before 0

(B's case with p $\neq$ q)

Probability (reaching 3 before 0| starting at 2)

$$= \frac{1-\left(\frac{q}{p}\right)^j}{1-\left(\frac{q}{p}\right)^N} = \frac{1-(3/2)^2}{1-(3/2)^3} = \frac{10}{19} \approx 0.5263$$

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