International Institute
for Technology and Management

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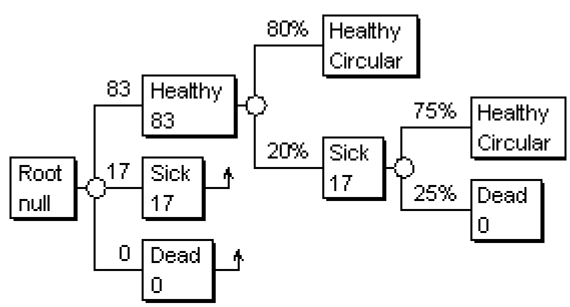
Unit 76: Management Mathematics

Handout #7c

Applications of Matrices III

Topic	Interpretation
Markov Chains Uses matrices to predict the changes between one stage and another in a dynamical model. The outcome of an experiment depends only on the outcome of the previous experiment.In other words, the next state of the system depends only on the present state not on preceding states. **In such a process, the past is irrelevant for predicting the future given knowledge of the present**. This type of processes is considered as mathematical models that evolve over time in a probabilistic manner and is a random function. In the most common applications, the domain over which the function is defined is a time interval (a process of this kind is called a *time series* in applications). Familiar examples of time series include stock market and exchange rate fluctuations, signals such as speech, audio and video; medical data such as a patient's blood pressure or temperature;	**Random variable** Unlike the common practice with other mathematical variables, a random variable cannot be assigned a value; a random variable does not describe the actual outcome of a particular experiment, but rather describes the possible, as-yet-undetermined outcomes in terms of real numbers. For example, a random variable can be used to describe the process of rolling a fair die and the possible outcomes {1, 2, 3, 4, 5, 6}. Another random variable might describe the possible outcomes of picking a random person and measuring his or her height. *Example1:* Assume you have a community of 100 people. Initially, 83 of the people are healthy and 17 are sick. You predict that each year 20% of the healthy people will get sick. Furthermore, 25% of the sick people will die and the remainder will get better. You want to know how many people will still be alive after 10, 20, 30, years. 

Transition Matrix

A transition matrix has the following features:

1. It is a square matrix, since all possible states must be used both as rows and as columns.

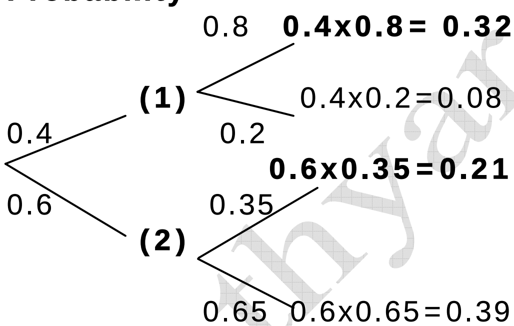
2. All entries are between *0 and 1*, inclusive, because all entries represent probabilities.

3. The sum of entries in any row must be **1**, because the numbers in a row give the probability of changing from the state at the left to one of the states indicated at the top.

Example3:

Suppose that when the campaign began, A had 40% of the market and B had 60%.Construct a probability tree and find how these proportions would change after another week of advertising.

Probability



Add the numbers indicated in **Bold** to find the portion of people taking their cleaning to A ,after one week : 0.32 + 0.21 = 0.53

Similarly the proportion taking their cleaning to B :

0.8 + 0.39 = 0.47

The initial distribution of 40% and 60% becomes after one week 53% and 47%.

These distributions can be written as the *probability vectors*:

[0.4 0.6] and [0.53 , 0.47]

Example2:

A small town has only two dry cleaners A and B. the manager of A hopes to increase the firm's market share by an extensive advertising campaign. After the campaign, a market research firm finds out that there is a probability of 0.8 that an A customer will use A's services and a 0.35 chance that a B's customer will switch to A's services.

If a customer bringing his load to A is said to be in **state 1** and a customer bringing load to B is said to be in **state 2**

Then these probabilities of change from one cleaner to the other are shown in the following table:

If there is 0.8 chance that an A customer will come back to A ,then there is

1- 0.8 =0.2 that the customer will switch to B.

If there is 0.35 chance that a customer will switch to A,then there is

1 – 0.35 = 0.65 that a customer will return to B.

		Second Load	
	State	**1**	**2**
First	**1**	0.8	0.2
Load	**2**	0.35	0.65

This can be represented by what is called a transition matrix where the states are indicated at the side and top as follows:

Load 2

A B

Load 1 $\begin{pmatrix} 0.8 & 0.2 \\ 0.35 & 0.65 \end{pmatrix}$

Probability Vectors

A probability vector is a matrix of only one row, having non-negative entries with the sum of entries equal to 1.

The results from the probability tree above are exactly the same as the result of **multiplying the initial probability vector by the transition matrix:**

$$(0.4 \quad 0.6)\begin{pmatrix} 0.8 & 0.2 \\ 0.35 & 0.65 \end{pmatrix} = (0.53 \quad 0.47)$$

<u>Remark:</u> To find the market share after two weeks, multiply the vector (0.53 0.47) by the transition matrix:

$$(0.53 \quad 0.47)\begin{pmatrix} 0.8 & 0.2 \\ 0.35 & 0.65 \end{pmatrix} = (0.59 \quad 0.41)$$

Equilibrium Vector

The equilibrium vectors gives a long-range-prediction-the shares of the market will stabilize (under the same conditions).

The probability vector can be found without doing all the work shown above.

By definition, V is the fixed probability vector (equilibrium probability vector) if VP = V

Where P is the transition matrix.

To find the equilibrium vector of the example above:

Let it be V(v_1 v_2) then :

$$(v_1 \quad v_2)\begin{pmatrix} 0.8 & 0.2 \\ 0.35 & 0.65 \end{pmatrix} = (v_1 \quad v_2)$$

$$(0.8v_1 + 0.35v_2 \quad 0.2v_1 + 0.65v_2) = (v_1 \quad v_2)$$

$0.8v_1 + 0.35v_2 = v_1$

$0.2v_1 + 0.65v_2 = v_2$

<u>Example4</u>:

The following table gives the market share (rounded) for each cleaner after various weeks:

Week	A	B
Start	0.4	0.6
1	0.53	0.47
2	0.59	0.41
3	0.62	0.38
4	0.63	0.37
5	0.63	0.37
…..	…..	…..
12	0.64	0.36

The results seem to approach the probability vector **(0.64 0.36)**. This vector is called the ***equilibrium vector*** or the fixed vector for the given transition matrix.

The equilibrium vectors gives a long-range-prediction-the shares of the market will stabilize (under the same conditions) at 64% for A and 36% for B.

<u>Remark:</u>

Starting with some other initial probability vector would give the **same** equilibrium vector. In fact ,**the long range trend is same no matter what the initial vector is.The long range trend depends only on the transition matrix not on the initial distribution.**

Knowing that $v_1 + v_2 = 1$, $v_1 = 1 - v_2$

Substituting into either of the two equations : $v_2 = 0.364 \approx 0.36$ and $v_1 = 0.636 \approx 0.64$

V = (0.64 0.36)