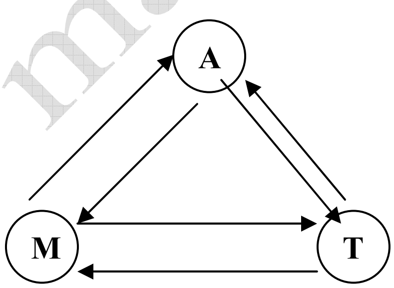
International Institute
for Technology and Management

January 24th, 2006

Unit 76: Management Mathematics **Handout #7b**

Applications of Matrices

Dowling ET ,Schaum's Outline series: Introduction to mathematical economics Chapters 12,pp. 259-260,276-279-Study Guide pp.43-46

Topic	Interpretation
Input-Output economy It uses a matrix representation of a nation's economy to predict the effect of changes in one industry on others,i.e. how changes in one economic sector may have an effect on other sectors. Input-output models are concerned with the production and flow of goods(and perhaps services). In an economy with **n** basic commodities (or **sectors**),the production of each commodity uses some(perhaps all) of the commodities in the economy as **inputs.** The amounts of each commodity used in the production of **1** unit of each commodity can be written as **nxn** matrix **A**, called the **Technology** matrix.  **Input-output model of 3 industries,Example1.**	<u>*Example1*</u>*:* Suppose a simplified economy involve just three sectors Agriculture(A), Manufacturing(M) and Transportation(T). The production of **1 unit of Agriculture** (One dollar's worth of Agriculture) requires the input of : \$0.2 worth of Agriculture \$0.4 worth of Manufacturing \$0.1 worth of Transportation. The production of **1 unit of Manufacturing** (One dollar's worth of Manufacturing) requires the input of : \$0.3 worth of Agriculture \$0.1 worth of Manufacturing \$0.3 worth of Transportation. The production of **1 unit of Transportation** (One dollar's worth of Transportation) requires the input of : \$0.2 worth of Agriculture \$0.2 worth of Manufacturing \$0.2 worth of Transportation. **The technology matrix :** **Output** **A M T** **Input** $\begin{pmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{pmatrix}$

Production schedule

The problem is to determine the production schedule which enables each process to meet all the demands for its product.
It is not obvious that it is possible to satisfy all the interlinked requirements, but we'll prove that under reasonable conditions, there is a unique solution.
Another matrix is used with the input-output model is a matrix giving the amount of each commodity produced (or required to meet all needs)called the **production matrix**.
In an economy of **n** commodities ,the production matrix is represented by a column matrix

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ x_n \end{pmatrix}$$

External Demand
In an **n**-commodity economy ,the external demand for the various commodities from outside the production system, is represented by a **demand** matrix **D:**

$$\mathbf{D} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ . \\ d_n \end{pmatrix}$$

Total demands
Is the external demand plus the quantity needed to produce each commodity.
See Example4

<u>Example2</u>:
In Example1 , if x_1 , x_2 and x_3 are the production levels required to satisfy all the demands in a given period ,the production matrix is :

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

<u>Example3</u>:

In Example1 ,suppose there is an external demand for 516 units(dollar's worth) of Agriculture , 258 units(dollar's worth) of Manufacturing and 129 units(dollar's worth) of Transportation.
The demand matrix :

$$\mathbf{D} = \begin{pmatrix} 516 \\ 258 \\ 129 \end{pmatrix}$$

<u>Example4</u>:

In Example1 , suppose there is an external demand of d_1 dollar's worth of Agriculture,d_2 dollar's worth of Manufacturing and d_3 dollar's worth of Transportation. if x_1 , x_2 and x_3 are the production levels required to satisfy all the demands in a given period
Total demand of Agriculture = d_1 + the quantity needed to produce Agriculture, Manufacturing and Transportation.
Each unit of Agriculture requires 0.2 units of A , each unit of M requires 0.3 units of A,each unit of T requires 0.2 of A
$x_1 = d_1 + 0.2x_1 + 0.3x_2 + 0.2x_3$
Similarly :
$x_2 = d_2 + 0.4x_1 + 0.1x_2 + 0.2x_3$
$x_1 = d_3 + 0.1x_1 + 0.3x_2 + 0.2x_3$

Finding the production levels

By solving the system :

X = D + AX

X – AX = D with **X = IX**

IX –AX = D

(I – A)X = D

X = (I – A)$^{-1}$ D

I – A is often known as the **Leontief** matrix.

Realistic economic conditions

In **Example5** ,we were able to find the inverse matrix **(I–A)$^{-1}$** thus guaranteeing a unique solution for any external demand **D**.Furthermore, the entries of **(I–A)$^{-1}$** turned out to be non-negative.

If X_i (i = 1,2,3,…..,n) is the total production of good i.

a_{ij} (i,j=1,2,3,….,n) is the proportion of every unit of good i produced ,consumed by industry j.

D_i (i=1,2,3,….,n) the final demand for the good i.

Then $X_i = \sum_{j=1}^{n} a_{ij} X_i + D_i$

It is reasonable to assume that the total $\sum_{j=1}^{n} a_{ij} X_i < 1$ because otherwise the production of X_i would make a loss.

Networks

Uses matrices to represent interrelationships for example transportation problems, flow between two points, allocation problems ,critical path(minimum completion time)for a project, optimal company communication.

Example5:

The technology matrix for a three-industry input-output model is :

$$A = \begin{pmatrix} 0.5 & 0 & 0.2 \\ 0.2 & 0.8 & 0.12 \\ 1 & 0.4 & 0 \end{pmatrix}$$

If the non-industry demand for the output of these industries is $d_1 = 5$,$d_2 = 3$ And $d_3 = 4$,determine the equilibrium output levels for these three industries.

If X is the output vector ,then

X = D + AX $\Rightarrow$ **X = (I-A)$^{-1}$D**

$$I-A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.5 & 0 & 0.2 \\ 0.2 & 0.8 & 0.12 \\ 1 & 0.4 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.5 & 0 & -0.2 \\ -0.2 & 0.2 & -0.12 \\ -1 & -0.4 & 1 \end{pmatrix}$$

$$(I - A)^{-1} = \begin{pmatrix} 7.6 & 4 & 2 \\ 16 & 5 & 5 \\ 14 & 10 & 5 \end{pmatrix}$$

$$X = (I-A)^{-1}D = \begin{pmatrix} 7.6 & 4 & 2 \\ 16 & 5 & 5 \\ 14 & 10 & 5 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 58 \\ 145 \\ 120 \end{pmatrix}$$

Therefore ,the necessary production amounts for the three commodities are 58, 145 and 120 units respectively.

Example6:

Roads connecting four cities :city1, city2, City3, city4.An example problem is how many ways are there from city1 to city2 by going through exactly one city?

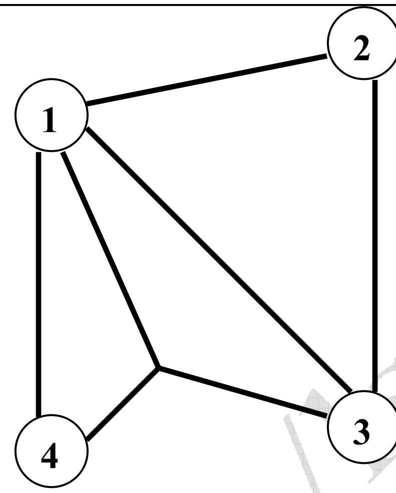
The figure shows the roads connecting four cities.This can be represented by a matrix **A** where the entries represent the number of roads connecting two cities without passing through another city. For example, there are two roads connecting city1 to city4 without passing though either city2 or city3.This information is entered in row1,column4 and again in row4,column1 of

matrix **A**= $\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$

a_{11} : roads from 1 to 1 = 0
a_{12} : roads from 1 to 2 = 1
a_{13} : roads from 1 to 3 = 2
a_{14} : roads from 1 to 4 = 2
and so on …….

A= $\begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{pmatrix}$

Note that there are 0 roads connecting each city to itself. Also, there is one road connecting cities 3 and 2.
How many ways are there from city1 to city2 by going through exactly one city?
Because we have to go from 1 to 2 through another city, we must go through either 3 or 4. From the diagram, we can go from 1 to 2 through 3 in 2 ways.



It is not possible to go from 1 to 2 through 4 because there is no direct route between 4 and 2.

The Matrix:
$\mathbf{A^2} = \mathbf{AA}$ (**A** multiplied by itself) gives the number of ways to travel between any two cities by passing through exactly one other city:

$$\mathbf{A^2} = \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 9 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \\ 3 & 2 & 6 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix}$$

Similarly, $\mathbf{A^3}=\mathbf{A^2A}$ gives the number of ways to travel between any two cities by passing exactly through two cities. Also $\mathbf{A} + \mathbf{A^2}$ gives the number of ways to travel between two cities with at most one intermediate city. The diagram may be given many other interpretations. For example mutual influence between people or communication of Telephone lines.