International Institute
for Technology and Management

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Unit 76: Management Mathematics

Handout #6a

Differential Equations I

Topic	Interpretation
Definition A differential equation expresses a relation between a function, its derivative and the independent variable. **Order** The order of the *highest derivative* present in the equation. **Degree** The highest *power* of the highest order of derivative present in the equation. **Solving differential equations** **a. Equations of the form y′ = f(t)** simply integrate both sides of the equation. Example 2: $\frac{dy}{dt} = e^t + 2$; $\int dy = \int (e^t + 2)dt$; $y = e^t + 2t + C$ **b. First order ,first degree Equation:** $P\frac{dy}{dt} + Q = 0$ Where P and Q are functions of y and t. **Case 1 : Separable variables** $\frac{dy}{dt} = f(y)g(t) \Rightarrow \frac{dy}{f(y)} = g(t)dt$ Then integrate both sides. ***(see Example 4)*** Example 5: $\frac{dy}{dt} = \frac{(y^2+5)e^{2t}}{2y}$; $y(0) = 1$	Examples1: (1) $\frac{dy}{dx} - 2y = e^x$ First order , first degree (2) $y'' + 4y' + 4y = t\ln t$ 2nd order ,1st degree (3) $\left(\frac{d^2y}{dt^2}\right)^3 - 2\left(\frac{dy}{dt}\right)^5 + y = t^2 + 1$ Second order ,third degree Example 3: Solve the D.E.(differential equation) $y'' = 6$ $y' = \int 6dt = 6t + c_1$ (note the c_1 because we need to integrate once more) $y = \int (6t + c_1)dt = 6t^2/2 + c_1t + c_2 = 3t^2 + c_1t + c_2$ Example 4: $\frac{dy}{dt} = \frac{2t+1}{y} \Rightarrow ydy = (2t+1)\,dt$ $\int ydy = \int (2t+1)dt \Rightarrow \frac{y^2}{2} = \frac{2t^2}{2} + t + c$ $\Rightarrow y = 2t^2 + 2t + 2c$ Let $2c = C \Rightarrow y = 2t^2 + 2t + C$ Actually no need to do all the arithmetic for the constants, just replace them by C. Example 6: $\frac{dp}{dt} = (4-p)^3 \Rightarrow \frac{dp}{(4-p)^3} = dt$; $\int \frac{dp}{(4-p)^3} = \int dt$; $u = 4 - p \Rightarrow du = -dp \Rightarrow \int \frac{-du}{u^3} = t + C$ $\int -u^3 du = t + C \Rightarrow ½\, u^{-2} = t + C$ $\Rightarrow ½\,(4-p)^{-2} = t + C$

$\Rightarrow 2y\dfrac{dy}{dt} = (y^2 + 5)e^{2t}$

$\Rightarrow \dfrac{2ydy}{y^2+5} = e^{2t}dt \Rightarrow \int \dfrac{2ydy}{y^2+5} = \int e^{2t}dt$

$Ln(y^2+5) = ½ e^{2t} + C$
$y(0) = 1$: substitute $t = 0$; $y = 1$
$\ln 6 = ½ + C \Rightarrow C = \ln 6 - ½$
$Ln(y^2+5) = ½ e^{2t} + \ln 6 - ½$

Case 2 :Homogeneous Equations:
An equation in x and y is said to be homogeneous if the sum of the powers of x and y is the same in all of the terms.
e.g. $x^2y^4 + 2x^5y - 3x^4y^2 + 4y^6$ is homogeneous of degree 6.
Solution method: Rewrite the equation: $P\dfrac{dy}{dt} + Q = 0$

$\dfrac{dy}{dt} = \dfrac{-Q}{P}$;divide both the numerator and denominator by x^n Where n is the degree of homogeneity, then set $v = \dfrac{y}{x}$

$y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Substitute this in the original equation to get a separable of variables equation in x and v .
(see Example 7)

Solving Equations using substitution :
1. Reducible to a homogeneous:

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

If $a_1b_2 - a_2b_1 = 0$ then use $a_1x + b_1y = t$;to get a separable variables

If $a_1b_2 - a_2b_1 \neq 0$; then use
$x = X + h$; $y = Y + k$
h and k are the solution of :
$a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$

<u>Example 7:</u>

$y^3 + (x^2y + x^3)\dfrac{dy}{dx} = 0$; homog. Of deg. 3

$\dfrac{dy}{dx} = \dfrac{-y^3}{x^2y + x^3}$ dividing both Num. & Den.

By x^3 : $\dfrac{dy}{dx} = \dfrac{-\left(\dfrac{y}{x}\right)^3}{\left(\dfrac{y}{x}\right)+1}$; set $v = \dfrac{y}{x}$

$y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$;substitute in the eq

$v + x\dfrac{dv}{dx} = \dfrac{-v^3}{v+1}$; separable :

$x\dfrac{dv}{dx} = \dfrac{-v^3 - v^2 - v}{v+1} \Rightarrow \dfrac{dx}{x} = -\dfrac{v+1}{v(v^2+v+1)}dv$

$\int \dfrac{dx}{x} = \int -\dfrac{v+1}{v(v^2+v+1)}dv$ **by partial fractions**

<u>Example8</u>: $\dfrac{dy}{dx} = \dfrac{2x+3y-7}{3x+2y-8}$

Let $x = X + 2$; $y = Y + 1$
(2 , 1) is solution of the system
$2x + 3y - 7 = 0$; $3x + 2y - 8 = 0$
The equation then becomes :

$\dfrac{dY}{dX} = \dfrac{2X+3Y}{3X+2Y}$ which is homogeneous degree 1

Now dividing both Num. & Den. by X and Letting $v = Y/X$ (*refer to Example 7):*

$2\dfrac{dX}{X} = \dfrac{2v+3}{v^2-1}dv$

Integrating and back substituting :
$(x - y - 1)^5 = C(x+y-3)$

2. yf(xy)dx + xf(xy)dy=0
Use $v = xy$; $y = v/x$ to get a separable variables Eq.

3. Other substitutions:
No general rule;the form of the equation leads you to choose the substitution;
e.g.: $(2+2x^2y^{1/2})ydx + (x^2y^{1/2}+2)xdy = 0$;
Set $v = x^2y^{1/2}$