International Institute
for Technology and Management

Unit 76: Management Mathematics

Handout #14

Course Summary

Chapter 1 : Set Theory

- $\cap$: ** means **AND ; $\cup$: means **OR** ; Compliment : means **NOT**.
Example: A = set of male students then A^c = set of female students.

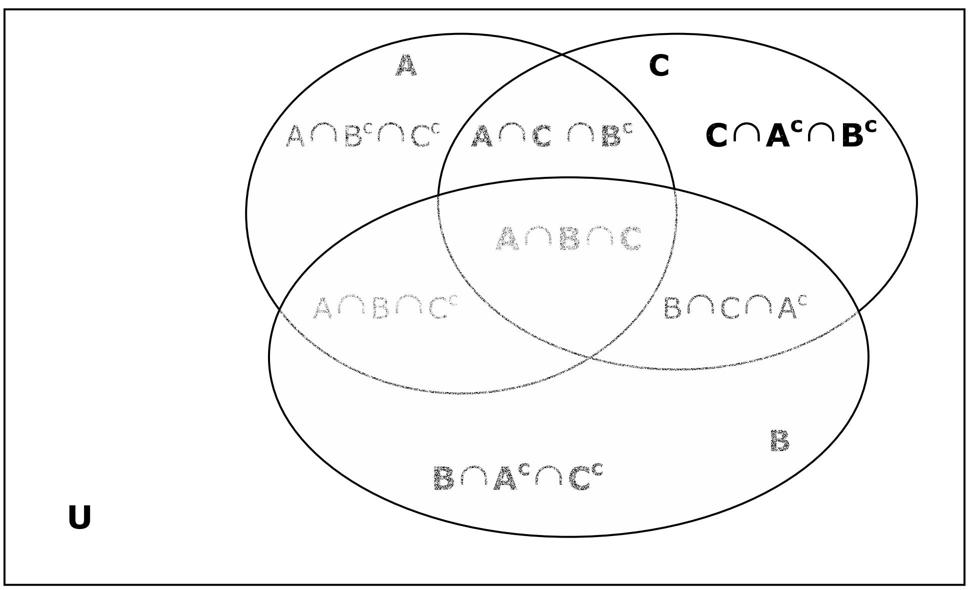
- Demorgan's Theorems: useful in describing sets with statements:

a. $(A\cap B)^c = A^c \cup B^c$
b. $(A\cup B)^c = A^c \cap B^c$

Example: set of people that are not (A: smokers) $\cap$(and) not (B: alcoholic) = $(A\cap B)^c$ = set of people that are (A^c = not smokers) or($\cup$)(B^c= (not alcoholic.) Using (a).

-Union rule for counting: $n(A\cup B)=n(A) + n(B)- n(A \cap B)$

-Venn Diagram:



-Discovering erroneous data:

1.When the sum of orders(number of elements) of all sets > the given number of observations.
2.When the order of one or more set is **negative**.

-Correcting the errors: Look at

1. The above 8 disjoint sets : try to decrease or increase their orders.
2. The orders of the original three sets: try to decrease or increase their orders
3. Usually ,don't tamper with sets of small order.

Chapter 2 : Index Numbers

- Simple price index (Single item): Price relative = $\frac{p_t}{p_0} \times 100$

P_t : price in a later period ; P_0 : price in the base period.

- Simple Aggregate Index:

1. Un weighted : $\frac{\sum_i^N p_{it}}{\sum_i^N p_{i0}} \times 100$; **2. Weighted :** $\frac{\sum_i^N p_{it} \times q_{it}}{\sum_i^N p_{i0} \times q_{i0}} \times 100$

- **Laspeyre's Price Index :** $\frac{\sum_i^N p_{it} \times q_{i0}}{\sum_i^N p_{i0} \times q_{i0}} \times 100$; **Paasche's Price index:** $\frac{\sum_i^N p_{it} \times q_{it}}{\sum_i^N p_{i0} \times q_{it}} \times 100$

Remark: Laspeyre's Index is ***base*** year weighted and Paasche's Price index is ***final*** year weighted and this is the main reason why they differ.

-Quantity(Volume) indices: Laspeyre's: $\frac{\sum_i^N p_{i0} \times q_{it}}{\sum_i^N p_{i0} \times q_{i0}} \times 100$; **Paasche's:** $\frac{\sum_i^N p_{it} \times q_{it}}{\sum_i^N p_{it} \times q_{i0}} \times 100$

The prices remain relatively stable and the quantities of items which are changing.

- Ideal Indices :

1. Irving Fischer Index*:* The geometric mean of the original indices= $\sqrt{\textbf{\textit{Laspeyer's}} \times \textbf{\textit{Paasche's}}}$ x 100

2. Marshall-Edgeworth Index: Uses the arithmetic mean of the quantities purchased in the base and the current periods as weights.

- Index Tests: Are used to determine how good an index is:

1. Time reversal test: Reversing the time subscripts produces the reciprocal of the original index: $\mathbf{I_{01}} \times \mathbf{I_{10}} = \mathbf{1}$; $\mathbf{I_{01}}$ **:** Index of current year with the base year 100; $\mathbf{I_{10}}$ **:** Index of the base year taking the current year as base.

2. Factor reversal test: The product of the price index and the quantity index should be equal to the index of total value.

-Chain base Index: the base period shifts for each successive index.

$$\text{Chain Base Index} = \frac{\textit{Index of current year}}{\textit{Index of previous year}} \times 100$$

- Splicing Index Numbers: Two or more overlapping series(say A and B) of Index numbers are **combined** into one series:

(Index of current year of B)x(Index of final year of A) / 100

Example: A:1990 to 1993 where I_{1993} = 130 ; B:1993 to 1996 where I_{1993} = 100 I_{1994} = 125 ; I_{1995} = 150 ; I_{1996} = 160 then I_{1994} = 125x130/100 = 162.5 etc….

-Shifting the base year: $\frac{\textit{Index of current year}}{\textit{Index of New base year}} \times 100$

-Deflating a Series :Real Prices = $\frac{\textit{current prices}}{\textit{Index of current year}} \times 100$

Chapter 3 : Trigonometric Functions

sin(-x) = -sin(x) cos(-x) = cos(x) tan(-x) = -tan(x) cot(-x) = -cot(x) sec(-x) = sec(x) csc(-x) = -csc(x)	sin(π/2-x) = cos(x), cos(π/2-x) = sin(x), tan(π/2-x) = cot(x), cot(π/2-x) = tan(x), sec(π/2-x) = csc(x), csc(π/2-x) = sec(x).	sin(π/2+x) = cos(x), cos(π/2+x) = -sin(x), tan(π/2+x) = -cot(x), cot(π/2+x) = -tan(x), sec(π/2+x) = -csc(x), csc(π/2+x) = sec(x).
sin(π-x) = sin(x), cos(π-x) = -cos(x), tan(π-x) = -tan(x), cot(π-x) = -cot(x), sec(π-x) = -sec(x), csc(π-x) = csc(x).	sin(π+x) = -sin(x), cos(π+x) = -cos(x), tan(π+x) = tan(x), cot(π+x) = cot(x), sec(π+x) = -sec(x), csc(π+x) = -csc(x).	**You don't have to memorize these, Use your Calculator to check them.** i.e. if you are faced with cos(t + π) choose t = 30 ;find cos 30=0.86 and cos (30+180)= cos210=-0.86;hence cos(t + 180) = -cost

Periodicity:
cos(α + 2kπ) = cosα ; sin(α + 2kπ) = sinα
e.g. cos(t + 6π) = cost ; sin(t + 5π) = sin (t + π + 4π) = sin(t+π) = - sint
tan(α + kπ) = tanα ; cot(α + kπ) = cotα
e.g. tan(t + 6π) = tant ; tan(t + 5π) = tant

Basic Relations :

$$\csc\alpha = \frac{1}{\sin\alpha} \qquad \sec\alpha = \frac{1}{\cos\alpha} \qquad \tan\alpha = \frac{1}{\cot\alpha} = \frac{\sin\alpha}{\cos\alpha}$$

$$\sin^2\alpha + \cos^2\alpha = 1 \qquad 1+\tan^2\alpha = \sec^2\alpha \qquad \sin\alpha = \pm\sqrt{1-\cos^2\alpha} \qquad \cos\alpha = \pm\sqrt{1-\sin^2\alpha}$$

Double Angle Relations

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha} \qquad \tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} \qquad \cot 2\alpha = \frac{\cot^2\alpha - 1}{2\cot\alpha}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$$

Exponential Relations

where: $i = \sqrt{-1}$; **Euler's Formula :** $e^{i\alpha} = \cos\alpha + i\sin\alpha$

$$\sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i} \qquad \cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2}$$

Derivatives:

y = sin ax	y' = a cosax	y = cos ax	y' = -a sinax
y = tanax	y' = a(1+tan²ax)	y =cotax	y' = -a(1+cot²x)

Integrals:

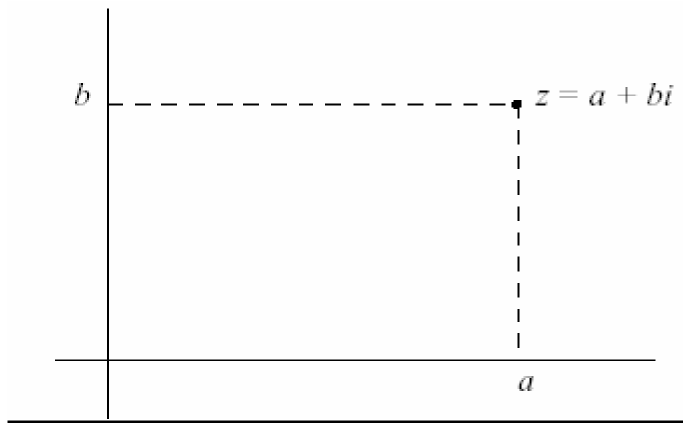
$$\int \sin ax\,dx = \frac{-1}{a}\text{cosax} + c \ ; \ \int \cos ax\,dx = \frac{1}{a}\text{sinax} + c$$

$$\int \tan ax\,dx = \frac{-1}{a}\ln(\text{cosax}) + c \ ; \ \int \cot ax\,dx = \frac{1}{a}\ln(\text{sinax}) + c$$

For comments, corrections, etc…Please contact Ahnaf Abbas: ahnaf@uaemath.com

Chapter 4 : Complex Numbers

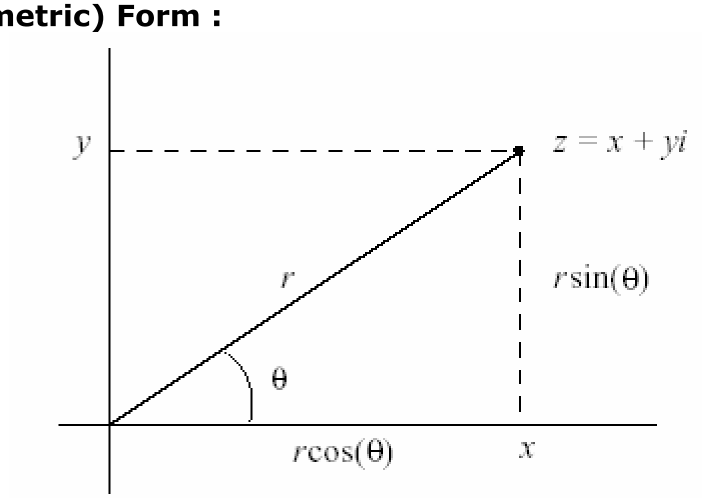
Algebraic Form : z = a + i b ; a = real part ; b = imaginary part ; $i^2 = -1$
Argand diagram :



Conjugate: z = a + ib ,the conjugate $\bar{z}$ = a − ib ; $z\bar{z} = a^2 + b^2$, for e.g.

$$\frac{2+i}{1+i} = \frac{2+i}{1+i}\frac{1-i}{1-i} = \frac{2-2i+i+1}{1+1} = \frac{3-i}{2} = \frac{3}{2} - \frac{1}{2}i.$$

Magnitude(Modulus): | z | = $\sqrt{z\bar{z}} = \sqrt{a^2+b^2}$
Polar (trigonometric) Form :



z = r(cosθ + isinθ) = re$^{i\theta}$; r = |z| = $\sqrt{x^2+y^2}$; tanθ = y/x ; θ =Argz (argument of z)[Recall Euler's : $e^{i\theta}$ = cosθ + isinθ]
e.g. z=1- i : r = $\sqrt{2}$ **;**tanθ = -1;θ = -π/4; z=$\sqrt{2}\,e^{-\pi/4i}$=$\sqrt{2}$[cos(-π/4)+isin(-π/4)]

Demiovre's : $(\cos\theta + i\sin\theta)^n = e^{in\theta}$ = cos(nθ) + isin(nθ)
z = r(cosθ + isinθ) ; $z^n = r^n(\cos\theta + i\sin\theta)^n = r^n e^{in\theta} = r^n[\cos(n\theta) + i\sin(n\theta)]$
e.g. $(-1 - i\sqrt{3})^6$: r = 2 ; θ = π/3 ;$[2(\cos(\pi/3)+i\sin(\pi/3))]^6$ =
2^6[cos6(π/3)+isin6(π/3)] = 64[cos2π+isin2π] = 64[1 + i(0)] = 64

Chapter 5 : Difference Equations

2^{nd} order linear difference equation: $y_{t+2} + ay_{t+1} + by_t = r_t$

Solution : $y_t = y_c + y_p$

Step1 : solve the homogeneous equation : $y_{t+2} + ay_{t+1} + by_t = 0$

The auxiliary equation : $r^2 + ar + b = 0$ **(Remember to arrange the equation from higher index to lowest i.e. y_{t+2} then y_{t+1} then y_t)**

Case 1 : r_1 and r_2 are real distinct: $y_t = Ar_1^t + Br_2^t$

Case 2 : r_1, r_2 are real and equal; $r = r_1 = r_2$: $y_t = (A + Bt)r^t$

Case 3 : r_1, r_2 are imaginary: $y_t = (\sqrt{b})^t (A\cos\alpha\, t + B\sin\alpha\, t)$; $\alpha = \cos^{-1}(\frac{-a}{2\sqrt{b}})$

Make sure your calculator mode is set to Radians.

Example: $9y_{t+2} + 6y_{t+1} + y_t = 2t + 1$; $y_1 = 1$, $y_2 = 3$

The auxiliary equation : $9r^2 + 6r + 1 = 0$ i.e. $(3r + 1)(3r + 1) = 0$

$r = -1/3$ two equal real roots. The complementary function : $y_c = (A + Bt)(-1/3)^t$

Step2 : Find the particular solution y_p according to the following table:

r_t	y_p	r_t	y_p
Constant C e.g. 6	C C	$a^t t^n$ e.g. $2^t (t)$	$a^t(A_0+A_1t+\dots+A_nt^n)$ $2^t(C + Dt)$
t^n e.g. $t + 3$ $2t^2$	$A_0+A_1t+\dots+A_nt^n$ $C + Dt$ $C + Dt + Et^2$	$a^t \sin bt$ e.g. $\sin 2t$	$a^t(A\cos bt + B\sin bt)$ $C\cos 2t + D\sin 2t$
a^t e.g. 2^t $2^t + t$	Ca^t $C2^t$ $C2^t + Dt + E$	$a^t \cos bt$ e.g. $\cos \pi t$	$a^t(A\cos bt + B\sin bt)$ $C\cos \pi t + D\sin \pi t$

Solution method: *Substitute the particular solution in the original equation to find the constants C & D .* **In the above example** : For a particular solution , $y_p = C + Dt$ substitute this in the original Equation : $9y_{t+2} + 6y_{t+1} + y_t = 2t + 1$

$\Rightarrow 9[C + D(t+2)] + 6[C + D(t+1)] + C + Dt = 2t + 1$; $D = 1/8$; $C = -1/8$

Hence $y_p = (1/8)t - 1/8$

Step3 : Write the general solution : $y_t = y_c + y_p$ and use the boundary conditions to find the constants A & B. **In the above example:**

$y_t = y_c + y_p = (A + Bt)(-1/3)^t + (1/8)t - 1/8$; with $y_1 = 1$, $y_2 = 3$,substitute ($t = 1; y = 1$) and ($t = 2$, $y = 3$) and solve for $A \approx -32$ and $B \approx -29$.

$y_t = (-32 + 29t)(-1/3)^t + (1/8)t - 1/8$

Behavior of the series:

General method: Find the limit of the general solution as $t \to \infty$:

-If the limit is ∞ ,then the series is divergent (unstable , explodes)

-If the limit is a *number* , then the series is convergent(stable)

Remember: $\lim_{t\to\infty} a^t = \infty$ **if $|a| > 1$ e.g. 2^∞** $\to \infty$

$\lim_{t\to\infty} a^t = 0$ **if $|a| < 1$ e.g. $(1/2)^\infty$** $= 0$

From the auxiliary roots: The auxiliary root with the largest absolute value is called the *dominant* root.

- Oscillation (part of the graph above the t-axis and the other part is below the t-axis) occurs when:

1. complex roots; **2.** Both roots are negative; **3.** the dominant root is negative.

-**Non Oscillating**:

1.Both roots are real **positive** ; 2. the dominant root is positive.

- **Stability** : Real roots :

1.The dominant root > 1 : $|r_1| > 1$: **Diverges.**

2.The dominant root < 1 : $|r_1| < 1$:**Converges.**

Complex roots :

1. $\sqrt{b}$ **> 1 : Diverges,** simply because it tends to ∞ this case(>1)

2. $\sqrt{b}$ **< 1 : converges,** simply because it tends to 0 this case(<1)

Examples:

$A(-1)^t + B(-6)^t + 3$ Both roots are negative :*Oscillating* |-6|> 1 : *Divergent*	$y_t = -4(1)^t + 6(10)^t - 3t$ Both roots are positive :Non *Oscillating* |10| > 1 : Divergent
$(\sqrt{2})^t(A\cos\frac{\pi}{2}t + B\sin\frac{\pi}{2}t) + 8$ Complex roots : *Oscillating* $\sqrt{2}$ **>** 1 : *Divergent*	$y_t = A(-3)^t + B(5)^t$ The dominant root = 5 is positive: Non *Oscillating* |5| > 1 : Divergent.

Remark: if there is a y_p ; then stability depends on the limit of y_p; see graph below!!

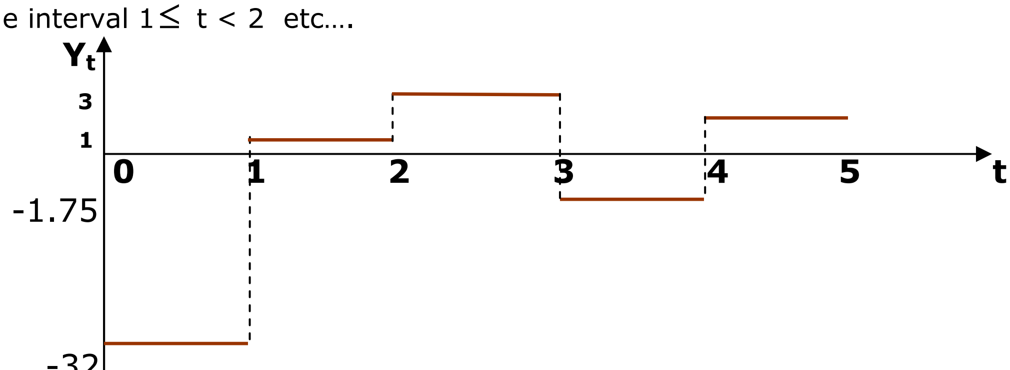
Graph: The graph of a Difference equation solution is always a **Step Graph :**

Example: the solution of the difference equation: $9y_{t+2} + 6y_{t+1} + y_t = 2t + 1$; $y_1=1$, $y_2= 3$ is given by : $y_t = (-32 + 29t)(-1/3)^t + (1/8)t - 1/8$

t	0	1	2	3	4
Y_t	-32	1	3	-1.75	1.4

t = 0 ; Y_t = -32 means, of the graph Y_t = -32 (str. line // t-axis) , we take only the part that lies in the interval $0 \leq t < 1$.

t = 1 ; Y_t = 1 means of the graph Y_t = 1 (str. line // t-axis) , we take only the part that lies in the interval $1 \leq t < 2$ etc….



Note that the graph is oscillating since both roots = -1/3 are negative.

Also you may think it is converging because |-1/3|< 1 or $(-1/3)^t$ --->0

It is divergent since the limit of yt = $(-32 + 29t)(-1/3)^t + (1/8)t - 1/8$ == 0 + ∞ - 1/8 = ∞

Chapter 6 : Differential Equations

First Order types :

1. Equations of the form y′ = f(t) ; simply integrate both sides of the equation.

2. Linear Equation $\frac{dy}{dt}$ + Py = Q **;** Where P & Q are functions of **t only**.

Solution : y= $e^{\int -Pdt}\left(\int Qe^{\int Pdt}\,\mathrm{dt}+c\right)$

3. Linear Equation P $\frac{dy}{dt}$ + Q = 0 ; Where P and Q are functions of **y** and **t**:

Case 1 : Separable variables $\frac{dy}{dt}=f(y)g(t)\Rightarrow\frac{dy}{f(y)}=g(t)dt$ Then integrate bothsides.

Case 2 : Homogenous equation:(sum of the powers of y and t is the same in all of the terms): P $\frac{dy}{dt}$ + Q = 0 ; $\frac{dy}{dt}$ = $\frac{-Q}{P}$;divide both the numerator and denominator by t^n ,Where n is the degree of homogeneity, then set y = vt ⇒ $\frac{dy}{dt}$ = v + t$\frac{dv}{dt}$;Substitute this in the original equation to get a separable equation of variables equation in t and v.

Second Order

$\frac{d^2y}{dt^2}$ + a$\frac{dy}{dt}$ + by = f(t) ; Similar situation to *Difference Equations.*

The general solution : $\mathbf{y = y_c + y_p}$

The auxiliary equation : $r^2+ar+b=0$ **(Remember to arrange the equation from higher index to lowest i.e. y″ then y′ then y)**

Case 1 : r_1 and r_2 are real distinct : $y_c = Ae^{r_1t}+Be^{r_2t}$

Case 2 : r_1, r_2 are real and equal; r = r_1= r_2 : $y_c = (A+Bt)e^{rt}$

Case 3 : r_1, r_2 are imaginary : $y_c = e^{\frac{-a}{2}t}(A\cos\alpha t+B\sin\alpha t)$ where $\alpha=\frac{\sqrt{4b-a^2}}{2}$ **(Because the solution depends on a and b; if the equation is given in the form : 2**y′′ + 6y′ +4y= 0 ;You need to divide by 2 to get the correct values of a and b : y′′ + 3y′+ 2y = 0).

Finding the Particular Solution:Use same table above (that of difference Eq.,p.5)

About Particular solutions

In some cases you may substitute the particular solution in the equation and get **no** values for the constants. This occurs when the **particular solution is part of the complementary function.** y′′ − 5y′+ 4y = e^{4x} ; $r^2-5r+4=0\Rightarrow$ r = 1 ; r = 4

$y_c = Ae^x+Be^{4x}$ **Note: e^{4x} is part of y_c: The particular solution $y_p = Ce^{4x}$ will not work!**

To fix it we attach x to Ce^{4x}:$y_p = Cxe^{4x}$

Examples:

$y'' + 3y' - 10y = 3t + e^{-5t} - 1$ The auxiliary roots : r = 2 , r = -5 $y_c = Ae^{2t} + Be^{-5t}$ The original particular solution: $y_p = C + Dt + Ee^{-5t}$ **will not work! Since** e^{-5t} is part of the complimentary function. **To fix it we attach t to Ee^{-5t}:** the correct one : $y_p = C + Dt + Ete^{-5t}$	$y'' - 9y = 5t^2e^{3t} + t\cos t - \sin t$ $y_c = Ae^{-3t} + Be^{3t}$ The original particular solution: $y_p = (C+Dt+Et^2)e^{3t} + (Ft+G)\cos t + (H+It)\sin t$ **will not work! Since** e^{3t} is part of the complimentary function. **To fix it we attach t to** $(C + Dt + Et^2)e^{3t}$**:** $y_p = t(C+Dt+Et^2)e^{3t} + (Ft+G)\cos t + (H+It)\sin t$
$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 28y = e^{-7x} + 7x - 1$ $y_c = Ae^{-7x} + Be^{4x}$ $y_p = Cxe^{-7x} + D + Ex$	$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 2e^{x}\cos x$ $y_c = e^x (A\cos x + B\sin x)$ $y_p = xe^x(C\cos x + D\sin x)$

Behavior of the series:

General method: Find the limit of the general solution as t ---→ ∞ :
-If the limit is ∞ ,then the series is divergent (unstable , explodes)
-If the limit is a *number* , then the series is convergent(stable)
Remember : As t--→ ∞ , e^{-t} -----→ 0 ; As t--→ ∞ , e^{t}--→ ∞

From the auxiliary roots:

Conditions for oscillation:
Assume the roots of the auxiliary equation are r_1 and r_2;time path is oscillating if both roots are **complex**.

Conditions for convergence:

1. **Two real distinct roots:** Both *negative* :$r_1 < 0$ and $r_2 < 0$: **Converges.**
   If one of the roots is *positive*: **Diverges.**
2. **Two real equal roots :** If the repeated root is *negative*: **Converges.**
   If the repeated root is *positive*: **Diverges.**
3. **Complex roots** : $e^{kt}(A\cos\alpha t + B\sin\alpha t)$; If k < 0 ; **converges.**

   Example : $y = e^{-3t}(2\cos 5t + 4\sin 5t)$: -3 < 0 ; it converges.**(**or e^{-3t} →0 as t→∞ **)**

Remark: if both roots are negative or the repeated root is negative and **there is a particular integral y_p** then the behavior depends on y_p on the long run.

Examples :

$y = 5e^{-t} - 3e^{-2t} + 7$ Both roots are negative: converges. **OR** As t-----→ ∞ **; e^{-t} -→ 0 ; e^{-2t} -→ 0** **y converges to 7 .**	$y = -2e^{3t} - e^{-2t} + 2$ One of the roots: 3 > 0:diverges. **OR** As t---→ ∞ **; e^{3t} -→ ∞ ; e^{-2t} -→ 0**
$y = e^{5t} - 3e^{2t} + t + 1$ Both roots are positive ;diverges. **OR** As t---> ∞ **; e^{5t} -→ ∞ ; e^{2t}-→ ∞**	$y = 5e^{-t} - 3e^{-2t} + t + 1$ As t-----→ ∞ **; e^{-t} -→ 0 ; e^{-2t} -→ 0** it depends on t + 1-→ ∞ **, diverges**

Chapter 7 : Applications on Matrices

Input-Output economy: Finding the production levels

By solving the system :

X = D + AX ; X = $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ x_n \end{pmatrix}$ **:** production levels **; D =** $\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ . \\ d_n \end{pmatrix}$ **:** the external demand for the various commodities from outside the production system; **A** : technology matrix ;

X = (I – A)$^{-1}$ D

I – A is often known as the **Leontief** matrix.

Example: The technology matrix for a three-industry input-output model is :

A = $\begin{pmatrix} 0.5 & 0 & 0.2 \\ 0.2 & 0.8 & 0.12 \\ 1 & 0.4 & 0 \end{pmatrix}$ If the non-industry demand for the output of these industries is $d_1 = 5$,$d_2 = 3$ and $d_3 = 4$,determine the equilibrium output levels for these three industries.

If X is the output vector ,then **X = D + AX** $\Rightarrow$ **X = (I-A)$^{-1}$D**

I-A = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.5 & 0 & 0.2 \\ 0.2 & 0.8 & 0.12 \\ 1 & 0.4 & 0 \end{pmatrix} = \begin{pmatrix} 0.5 & 0 & -0.2 \\ -0.2 & 0.2 & -0.12 \\ -1 & -0.4 & 1 \end{pmatrix}$

(I – A)$^{-1}$ = $\begin{pmatrix} 7.6 & 4 & 2 \\ 16 & 5 & 5 \\ 14 & 10 & 5 \end{pmatrix}$; **X = (I-A)$^{-1}$D =** $\begin{pmatrix} 7.6 & 4 & 2 \\ 16 & 5 & 5 \\ 14 & 10 & 5 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 58 \\ 145 \\ 120 \end{pmatrix}$

Therefore ,the necessary production amounts for the three commodities are 58, 145 and 120 units respectively.

Connectivity Matrices:

A= $\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$ **;** a_{11} : roads from 1 to 1;a_{12} : roads from 1 to 2 and so on …….

A^2 = AA (**A** multiplied by itself)gives the number of ways to travel between any two nodes by passing through exactly one other city

Similarly, **A^3=A^{2}A** gives the number of ways to travel between any two nodes by passing exactly through two cities. **A + A^2** gives the number of ways to travel between two nodes with at most one intermediate city.

Chapter 8 : Markov chains and Stochastic processes

The Gambler's Ruin

Gambler **A** has **$ j ;** Gambler **B** has **$ N –j** ; Total = **$ N** between them
Bet **$1** at a time(timid strategy); Gambler **A** wins with probability ***p***;Gambler **A** loses with probability ***q=1−p ;***

Case 1:* two absorbing barriers*: Game ends when 0 or N is reached.

Pr(Reaching N before 0 : winning all) $= \dfrac{\left(\frac{q}{p}\right)^{j} - \left(\frac{q}{p}\right)^{N}}{1-\left(\frac{q}{p}\right)^{N}}$ **if p ≠ q**

$= \dfrac{N-j}{N}$ **if p = q**

Pr(Reaching 0 before N : loosing all) = $\dfrac{1-\left(\frac{q}{p}\right)^{j}}{1-\left(\frac{q}{p}\right)^{N}} \quad if \ \ p \neq q$

$= \dfrac{j}{N}$ if p = q

***Case 2 :* One absorbing barrier at 0 :**

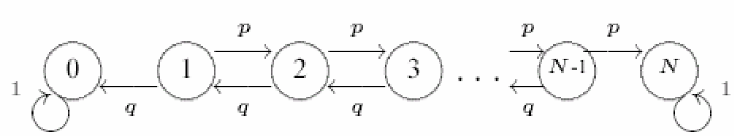
Pr(winning all)= $\begin{cases} \left(\frac{q}{p}\right)^{j} & if \ q < p \\ 1 \ \ if \ q \geq p & \end{cases}$

Pr(loosing all)= $\begin{cases} \left(\frac{q}{p}\right)^{j} & if \ q < p \\ 1 \ \ if \ q \geq p & \end{cases}$

For daring strategy refer to Assignment #6 ,questions 8 and 9.
Gambler's ruin is a simple random walk :

Transition Diagram for

Gambler's Ruin



Markov chains

$P(X_{t+1} = s \mid X_t = s_t ; X_{t-1} = s_{t-1} ;\dots; X_1 = s_1; X_0 = s_0) = P(X_{t+1} = s \mid X_t = s_t)$
for all $t = 1; 2; 3;$ ……. and for all states $s_0; s_1$; ……… ; s_t ; s.

Properties of the Transition Matrix

1. It is a square matrix **(n x n)**
2. All entries are between ***0** and **1***, inclusive, $0 \le p_{ij} \le 1$ because all entries represent probabilities.
3. The sum of entries in any row must be **1** .
4. Entry (i;j) is the conditional probability that **NEXT= j** , given that **NOW= i** : $p_{ij} = P(X_{t+1} = j \mid X_t = i)$
5. **P** is the **one step** transition matrix and P^2 is the **two step** transition matrix. In other words, the p_{12} entry in P^2 is the **probability of going from state1 to state 2 in two steps.**

   In general : $\Pr(X_n = j \mid X_0 = i) = P_{ij}^n$

   The probability that the chain is in state **j** at time **t = n** given that the initial state(at time t = 0) is state **i** = the **ij**th entry of P^n

<u>**Important**</u>**: The Meaning of the entries:**

-Every entry represents moving from one state to another in **one step**.

- A step means a **time** ,if $p_{12} = 0.2$ then this means it is possible to go from state 1 to state 2 at time **t = 1** (in one step) with a probability of 0.2.
- Every entry represent a **direct path** , if $p_{21} = 0$,then this means it is **not** possible to go from state 2 to state 1 directly at **t = 1** .However it ***may*** be possible to go from state 2 to state 1 in more than one step at t = 2,3,etc.

The Chapman-Kolmorgorove Equations

Suppose p^0 is the row vector of probabilities of the initial states(states at t = 0) p^n is the row vector of probabilities of the states at t = n ;then the general Markov chain at states n = 1,2,3,….. :

1. $p^n = p^0 p^n$
2. If p^n tends to a limiting distribution π ; then $\pi P = \pi$

Markov Processes: These differ from Markov chains in one major aspect: time now becomes a continuous quantity (i.e. **now it is time as we know it)**, The simplest example of a Markov process is the so called **Poisson**.

The poisson process

It expresses the probability of a number of events occurring in a fixed time if these events occur with a known average rate, and are independent of the time since the last event. A number of discrete occurrences (sometimes called "arrivals") that take place during a time-interval of given length. The probability that there are exactly *k* occurrences is a poisson distribution of **rate λ**:

$$\mathbf{P}(\lambda) = \frac{e^{-\lambda}\lambda^k}{k!}$$

λ is a positive real number, equal to the expected number of occurrences that occur during the given interval. For instance, if the events occur on average every 4 minutes, and you are interested in the number of events occurring in a 10 minute interval, you would use as model a Poisson distribution with λ = 2.5.

Examples: Calls arriving at a telephone exchange in a day.

The number of people joining a queue in an hour.

A stochastic process *N*(*t*) ; t≥ 0 is a (time-homogeneous, one-dimensional) Poisson process if,
-The number of events occurring in two disjoint (**non-overlapping**) subintervals are independent random variables*(see No. 2 in the assumptions below).*Arrivals are *memoryless* i.e. independent of what has happened before.
–Pr(k arrivals in time t) = P(λt) = $\frac{e^{-\lambda t}(\lambda t)^k}{k!}$;

i.e. a Poisson distribution of parameter λt .

Assumptions of the Poisson process:

If {N(t) ; t≥ 0 } is poisson process of parameter λ then :

1. N(0) = 0
2. For any $t_0 = 0 < t_1 < t_2 < \dots < t_n$,the process increments
   $N(t_1) - N(t_0)$: the number of events in $(0,t_1]$
   $N(t_2) - N(t_1)$: the number of events in $(t_1,t_2]$
   etc……
   are independent random variables.
3. The number of events occurring in the time interval (s , t +s] is a Poisson(λt); i.e. Pr[N(s+t) − N(s)] ~ P(λt)
4. N(t) ~ P(λt) ; i.e. Pr(k arrivals in time t) = P(λt)

Example:

A company expects on average,four of its trucks will break down in a one-month Period.Assuming a Poisson distribution is appropriate,what is the probability that exactly four trucks break down in a month? in a two month period?

If X is the number of trucks which break down in a month then as given

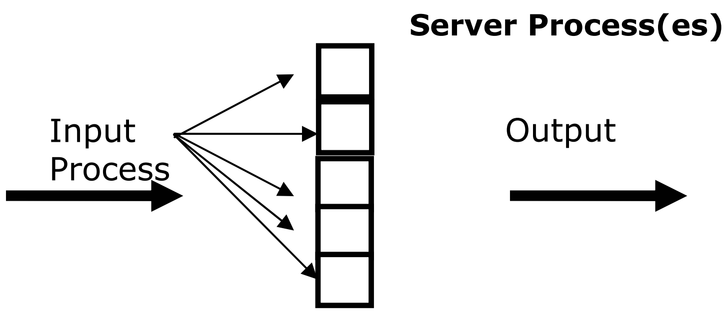
X ~ P(4) = $\frac{e^{-\lambda}\lambda^{k}}{k!} = \frac{e^{-4}4^{4}}{4!}$ = 0.195 i.e. 19.5 % of months would have 4 trucks breakdown.

If the rate of breakdowns in a month is 4 ,then the rate of breakdowns in two months is 2x4 = 8 ; hence the No. of breakdowns in two months ~ P(8)

= $\frac{e^{-\lambda}\lambda^{k}}{k!} = \frac{e^{-8}8^{4}}{4!}$ **= 0.0572**

Queuing Theory

Queuing Theory deals with systems of the following type:



Typically we are interested in how much queuing occurs or in the delays at the servers.

A standard notation is used in queuing theory to denote the type of system we are dealing with.

Typical examples are:

- M/M/1 Poisson Input/Poisson Server/1 Server
- M/G/1 Poisson Input/General Server/1 Server
- D/G/n Deterministic Input/General Server/n Servers
- E/G/∞ Erlangian Input/General Server/Infinite Servers

The *first* letter indicates the *input process*(**M = Memoryless = Poisson**) , the *second* letter is the *server* process and the *number* is the **number of servers**.

The simplest queue is the M/M/1 queue

Memoryless=Poisson/Memoryless=Poisson/1 server

Chapter 9 : Applications of Calculus

Taylor's Expansion

The Taylor polynomial for the function f(x) about x=a is

$$f(x)=f(a)+\frac{x-a}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+.......$$

Maclaurin's Expansion

With a = 0 , $f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+.......$

Famous Expansions :

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$; $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$ ....

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$ ....; $\ln(a+x) = \ln a + \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3}$ -............

Deducing Expansions

$e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + ... = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + ...$

$\cos x - 1 = -\frac{x^2}{2!} + \frac{x^4}{4!} - ......$ Now replace the whole expansion of (cosx−1) by x in the expansion of e^x :

$$e^{\cos x-1} = 1 + \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - ...\right) + \frac{1}{2!}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - ...\right)^2 + \frac{1}{4!}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - ...\right)^4$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{6} + ...\text{(only up to } x^4\text{)}$$

Note : for the square : find the first two terms only as in $(a-b)^2 = a^2 - 2ab$;for the Cube and up : cube only the first term . **You can always take $e^{\cos x-1}$ as new function and find its expansion**.

Simpson's rule : is used to approximate definite integrals:

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[f(a)+4f(a+h)+2f(a+2h)+4f(a+3h)......+f(b)\right]$$

FETO : Four times even ordinates ; two times odd ordinates.

Simpson's rule with ***n*** ordinates: $h = \frac{b-a}{n-1}$.

The result of Simpson's should be exactly equal to the result obtained by normal integration.

Consumers & Producers surpluses

$$CS = \int_0^q P^D dq - pq \quad ; \quad PS = pq - \int_0^q P^S dq$$

Where p and q are Equilibrium price and quantity respectively.

Chapter 10 : Optimization

Constrained Optimization:

Suppose f(x,y,z,…) has to be minimized or maximized subject to the constraint g(x,y,z,…) = 0 .

1. Using substitution:

Example : Minimize $f = x^2 + 4y^2$ subject to the constraint x - y = 10
Express one of the variables in terms of the others : y = x - 10
Substitute in the objective function : $f = x^2 + 4(x-10)^2 = 5x^2 - 80x + 400$
$f'(x) = 0 \Rightarrow 10x - 80 = 0 \Rightarrow x = 8$ & $y = x - 10 = -2$; (x,y) = (8 , -2).
When substitution is difficult to solve ,we use the method of

2. The Lagrange Multipliers:

1.)Set: $L(x,y,z,\ldots\lambda) = f(x,y,z,\ldots) - \lambda g(x,y,z,\ldots)$

2.)Find x and y as solutions of:

$$\frac{\partial L}{\partial x} = 0 ,\ \frac{\partial L}{\partial y} = 0 ,\ \frac{\partial L}{\partial z} = 0, \ldots,\ \frac{\partial L}{\partial \lambda} = g(x,y,z,\ldots) = 0$$

Multiple constraints

Suppose f(x,y,z,…) has to be minimized or maximized subject to the constraint $g_1(x,y,z,\ldots) = 0$, $g_2(x,y,z,\ldots) = 0$, $g_3(x,y,z,\ldots) = 0$,…… Then use :

$$L(x,y,z,\ldots\lambda) = f(x,y,z,\ldots) - \lambda_1 g_1(x,y,z,\ldots) - \lambda_2 g_2(x,y,z,\ldots) - \ldots\ldots$$

$$\frac{\partial L}{\partial x} = 0 ,\ \frac{\partial L}{\partial y} = 0 ,\ \frac{\partial L}{\partial z} = 0,\ldots,\ \frac{\partial L}{\partial \lambda_1} = g_1(x,y,z,\ldots) = 0 ,\ \frac{\partial L}{\partial \lambda_2} = g_2(x,y,z,\ldots) = 0 ,\ldots$$

Meaning of the multiplier

The Lagrange multiplier is an *artificial* variable added for *computational convenience*,suppose the optimum value is at the point **P(x,y)** , the constraint function **g(P)** can be thought of as "competing" with the desired function **f(P)** to "pull" the point **P** to its minimum or maximum. The Lagrange multiplier **λ** can be thought of as a measure of how hard **g(P)** has to pull in order to make those "forces" balance out on the constraint surface.

Economic Interpretation

The economic interpretation of the multiplier as a 'shadow price'.
The values λ have an important economic interpretation: If the right hand side b of Constraint *i* is increased by Δ, then the optimum objective value increases by approximately $\lambda\ \Delta$.