International Institute for Technology and Management

November 29th, 2007 **Unit: 05a – Mathematics 1**

GROUP(B)-VERSION B

This paper is not to be removed from the Examination Halls

SOLUTION

Student Name :

Student Number :

Tuesday 27th November 13 : 30 pm – 15 : 30 pm

Candidates should answer **NINE** of the following **ELEVEN** questions: **SEVEN** from section A (60 marks in total) and **TWO** from section B (20 marks each).

Candidates are strongly advised to divide their time accordingly.

Graph paper is provided at the student request.

Calculators **May NOT** be used for this paper.

PLEASE TURN OVER

SECTION A

Answer all **SEVEN** questions from this section (60 marks in total)

1. The supply equation for a good is

$$\mathbf{q = p^2 - 2p + 3}$$

and the demand equation is

$$\mathbf{q = -2p^2 + p + 9}$$

where p is the price.

Sketch the supply and the demand functions for $\mathbf{p} \geq 0$

Determine the equilibrium price and quantity.

Supply $\mathbf{q = p^2 - 2p + 3}$

(1)It has U shape since it has positive p^2 term

(2)Intercepts: <u>p-intercepts</u> : q = 0

$p^2 - 2p + 3 = 0$, $b^2 - 4ac = 4 - 4(1)(3) = -8 < 0$

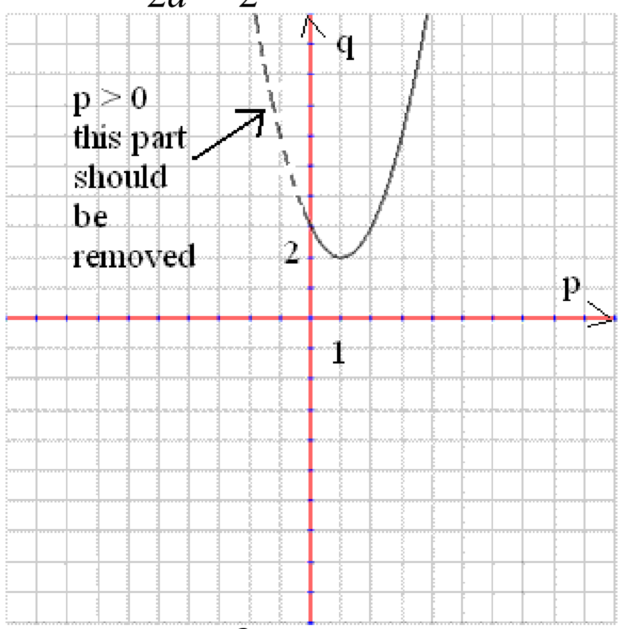
$\Rightarrow$ No intersection with the p-axis

<u>q-intercpt</u>: p = 0 $\Rightarrow$ q = 3 ; (0,3)

(3) The minimum : $q' = 2p - 2 = 0 \Rightarrow p = 1$

$\Rightarrow$ q = 2 ; (1 , 2)

OR $p = \dfrac{-b}{2a} = \dfrac{2}{2} = 1 \Rightarrow q = 2 \Rightarrow V(1,2)$



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The demand : $\mathbf{q = -2p^2 + p + 9}$

(1) It has $\cap$ shape since it has negative p^2 term .

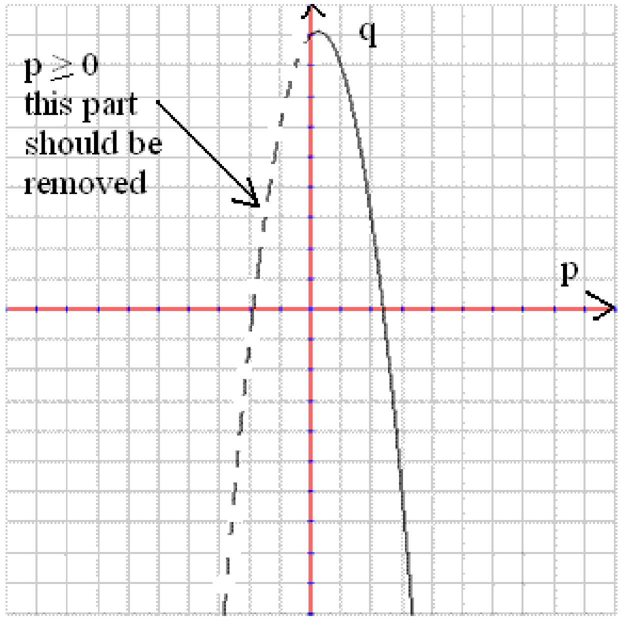
(2) Intercepts: <u>p-intercept</u> , q = 0 $\Rightarrow -2p^2 + p + 9 = 0$

$$p = \frac{-1 \pm \sqrt{73}}{2} \Rightarrow \left(\frac{-1-\sqrt{73}}{2}, 0\right) \text{ and } \left(\frac{-1+\sqrt{73}}{2}, 0\right)$$

<u>q-intercepts</u> : p = 0 $\Rightarrow$ q = 9 ; (0 , 9)

(3) The maximum : q′ = -4p + 1 = 0 ⇒ $p = \frac{1}{4}$

$q = \frac{73}{8}$ OR p = $\frac{-b}{2a} = \frac{1}{4} \Rightarrow q = \frac{73}{8} \Rightarrow$ V($\frac{1}{4}, \frac{73}{8}$)



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To determine the equilibrium price, we solve:
$p^2 - 2p + 3 = -2p^2 + p + 9 \Rightarrow 3p^2 - 3p - 6 = 0$
which is equivalent to $p^2 - p - 2 = 0 \Rightarrow (p+1)(p-2)=0$ 4
Either p = - 1 or p=2 of which only p=2 is economically
Meaningful. p=2 ⇒ q = 3

2. Find the maximum value of the function, verify that it is a maximum:

$$f(x) = (x^2 - x + 1)e^{-x}$$

$f'(x) = (2x-1)e^{-x} - (x^2 - x + 1)e^{-x} = (-x^2 + 3x - 2)e^{x}$

$f'(x) = 0 \Rightarrow -x^2 + 3x - 2 = 0$ (Since $e^x > 0 \ \forall x$)

⇒ either x = 1 or x = 2 7

$f''(x) = (-2x+3)e^{-x} - (-x^2 + 3x - 2)e^{-x} = (x^2 - 5x + 5)e^{-x}$

$f''(2) = -e^{-2} < 0 \Rightarrow x = 2$ maximises f(x).

3. Determine the following integrals $\int \frac{(1-\ln x)^2}{x}\,dx$

By substitution: $t = 1 - \ln x \Rightarrow dt = \frac{-1}{x}\,dx$ 5

$$\int \frac{(1-\ln x)^2}{x}\,dx = -\int t^2\,dt = -t^3/3 + C = -(1-\ln x)^3/3 + C$$

$\int \sin^3 x \cos^5 x dx$

By substitution t = cosx $\Rightarrow$ dt = - sinx dx $\Rightarrow$ dx = - dt/sinx

$\int \sin^3 x \cos^5 x dx = -\int \sin^3 x t^5 \frac{dt}{\sin x} = -\int \sin^2 x t^5 dt$

But $\sin^2 x = 1 - \cos^2 x = 1 - t^2$

$= -\int (1-t^2)t^5 dt = -\int (t^5 - t^7)dt = -t^6/6 + t^8/8 + C$

$= -\frac{\cos^6 x}{6} + \frac{\cos^8 x}{8} + C$ 5

4. The marginal cost for a company is

$4q^3 + 6q + e^q - 1$

and fixed costs of 60 .

Find the total cost, the variable cost and the average cost functions.

TC = $\int MCdq = \int (4q^3 + 6q + e^q - 1)dq$

TC = $q^4 + 3q^2 + e^q - q + C$

FC = TC(0) = 60 $\Rightarrow$ $0 + 0 + e^0 - 0 + C = 60$ $\Rightarrow$ 1+C =60

C = 59

TC = $q^4 + 3q^2 + e^q - q + 59$ 3

TC = VC + FC $\Rightarrow$ VC =TC – FC= $q^4 + 3q^2 + e^q - q + 59 - 60$

VC = $q^4 + 3q^2 + e^q - q - 1$ 2

AC = $\frac{TC}{q} = q^3 + 3q + \frac{e^q}{q} - 1 + \frac{59}{q}$ 2

5. (a) A firm's demand function is

p = aq + b (a < 0 ; b > 0)

Fixed costs are ***c*** and variable costs are ***d*** per unit.

Show that the profit is maximized when $q = \frac{d-b}{2a}$

The profit π = TR – TC

TR = p x q and TC = VC + FC = ***dq + c***

Now p = aq + b $\Rightarrow$ TR = (aq + b) x q= $aq^2 + bq$

$\pi = TR - TC = aq^2 + bq - dq - c = aq^2 + (b-d)q - c$

$\pi' = 2aq + b - d = 0 \Rightarrow q = \frac{d-b}{2a}$ 6

$\pi'' = 2a < 0$ (since a < 0) $\Rightarrow q = \frac{d-b}{2a}$ maximises the profit.

6. A firm has average variable cost

$$q^2 + 7q + \frac{\ln(q^3+7)}{q}$$

and fixed costs of 7 .Find the total cost function and the marginal cost function.

VC = q x AVC = $q^3 + 7q^2 + \ln(q^3 + 7)$
TC = VC + FC = $q^3 + 7q^2 + \ln(q^3 + 7) + 7$

MC = (TC)′ = $3q^2 + 14q + \dfrac{3q^2}{q^3+7}$ 6

7. Determine the following integrals

$$\int \frac{1}{x\sqrt{\ln x}(\ln x + 4\sqrt{\ln x} + 4)}dx \quad , t = \sqrt{\ln x} \Rightarrow dt = \frac{dx}{2x\sqrt{\ln x}}$$

$$2\int \frac{1}{(t^2+4t+4)}dt = 2\int \frac{1}{(t+2)^2}dt = 2\int (t+2)^{-2}dt$$

$$= \frac{-2}{t+2} + C = \frac{-2}{\sqrt{\ln x}+2} + C$$ 5

$$\int \frac{\sqrt{1+\tan x}}{\cos^2 x}dx \quad , t = 1 + \tan x \Rightarrow dt = dx / \cos^2 x$$

$$= \int \sqrt{t}dt = \int t^{1/2}dt = \frac{2t^{3/2}}{3} + C = \frac{2(1+\tan x)^{3/2}}{3} + C$$ 5

SECTION B

Answer **TWO** questions from this section (20 marks each)

8. (a) A firm is a monopoly for the good it produces, It has a marginal cost function MC = $6q^2 + 8$ and fixed costs of 20.The demand equation for its good is given by $p + 2q = 40$ where p is the price. Find expressions in terms of q, for the total revenue and profit. Determine the value of q that maximises the profit:

TR = pxq , but $p + 2q = 40 \Rightarrow p = 40 - 2q$ 2
TR = (40 -2q)xq = $40q - 2q^2$

TC = $\int MCdq = \int (6q^2+8)dq = 2q^3 + 8q + C$ 2
FC=TC(0) = 20 $\Rightarrow$ 0 + 0 + C = 20 , C = 20
TC = $2q^3 + 8q + 20$

π = TR – TC = $40q - 2q^2 - 2q^3 - 8q - 20 = -2q^3 - 2q^2 + 32q - 20$ 2
$\pi' = -6q^2 - 4q + 32 = 0 \Rightarrow -3q^2 - 2q + 16 = 0$

q = $\dfrac{2 \pm \sqrt{4-4(-3)(16)}}{-6}$ = $\dfrac{2 \pm \sqrt{4+192}}{-6}$ = $\dfrac{2 \pm \sqrt{196}}{-6}$

q = $\dfrac{2 \pm 14}{-6}$ $\Rightarrow$ q = 2 , q = - 16/6 = - 8/3 which is economically not feasible , therefore q = 2 **2**

π'' = -12q – 4 , π'' (2) = -12(2) – 4 = - 28 < 0 $\Rightarrow$ q = 2 maximises **2**
The profit.

(b) Determine the following integrals

$\int x^2\sqrt{x^3+1}\,dx$, t = x^3 + 1 $\Rightarrow$ dt = $3x^2$ dx $\Rightarrow$ x^2dx = dt/3

$\int \sqrt{t}\,dt/3 = \dfrac{1}{3}\int t^{1/2}dt$ = $\dfrac{1}{3} \times \dfrac{2t^{3/2}}{3} + C$ = $\dfrac{2(x^3+1)^{3/2}}{9} + C$ **5**

$\int \dfrac{x}{\sqrt{x+1}}\,dx$, t^2 = x + 1 $\Rightarrow$ 2t dt = dx

$\int \dfrac{x}{\sqrt{t^2}} \times 2\text{t}dt$ but, t^2 = x + 1 $\Rightarrow$ x = t^2 – 1

$$\int \frac{t^2-1}{t} \times 2\text{t}dt = 2\int (t^2-1)dt = \frac{2t^3}{3} - 2t + C$$

Now t^2 = x + 1 $\Rightarrow$ t = $\sqrt{x+1}$

$\dfrac{2t^3}{3} - 2t + C$ = $\dfrac{2(\sqrt{x+1})^3}{3} - 2\sqrt{x+1} + C$ **5**

9. (a) A firm is a monopoly its fixed costs are 20 it has average variable cost function ***AVC = 10 + q*** where q denotes its production level , the demand function of the good produced by firm is ***q = 10 -*** $\dfrac{p}{2}$ where p is the price. Find expressions, in terms of q , for the revenue and the profit and determine the value of q that maximizes the profit. Calculate this maximum profit.

TR = pxq , but ***q = 10 –*** p/2 $\Rightarrow$ p = 20 – 2q **2**
TR = (20 - 2q)xq = 20q – $2q^2$
TC = VC + FC = q x AVC + FC = 10q + q^2 + 20 **2**
π = TR – TC = 20q – $2q^2$ - 10q - q^2 - 20 = - $3q^2$+10q -20
π' = -6q+10 = 0 $\Rightarrow$ q = 10/6 = 5/3 **2**
π'' = -6 < 0 , q = 5/3 maximises the profit.
Value of the maximum : π(5/3) = - $3(5/3)^2$+10(5/3) -20 = -85/3 **2**

(b) The function $\boldsymbol{f(x) = x^2 - \ln(\sqrt{2}\,x)}$ is defined for $x > 0$
Determine the critical points of f and specify their nature.

> F (x) = ln (cx)
> $\Rightarrow F'(x) = \frac{1}{cx} \times c = \frac{1}{x}$
> Example: $f(x) = \ln \sqrt{2}\,x \Rightarrow f'(x) = 1/x$

2	$f'(x) = 2x - \frac{1}{x} = \frac{2x^2-1}{x} = 0$
2	$\Rightarrow 2x^2 - 1 = 0 \Rightarrow x^2 = ½$
2	$\Rightarrow x = \pm \frac{1}{\sqrt{2}}$ $x = -\frac{1}{\sqrt{2}}$ rejected since it doesn't belong to the domain of f.

[2] $x = \frac{1}{\sqrt{2}} \Rightarrow f(x) = ½ - \ln 1 = ½$ the critical point $(\frac{1}{\sqrt{2}}, \frac{1}{2})$

[2] $f''(x) = 2 + \frac{1}{x^2} > 0$ for every $x \Rightarrow x = \frac{1}{\sqrt{2}}$ minimises f

10.(a) A firm's marginal revenue function is $MR = 11 - q$
The firm's marginal cost function is
$MC = q^2 - 3q + 3$
where q is either the quantity sold or produced.
Find the profit-maximizing level of output and verify that it is a maximum.

$TC = \int (q^2 - 3q + 3)dq = q^3/3 - 3q^2/2 + 3q + C$ [2]

$TR = \int (11 - q)dq = 11q - q^2/2$ [2]

$\pi = TR - TC = 11q - q^2/2 - q^3/3 + 3q^2/2 - 3q - C$ [2]

$\pi = -q^3/3 + q^2 + 8q - C \Rightarrow \pi' = -q^2 + 2q + 8 = 0$ [2]

q = - 2 which is economically not feasible, q = 4

$\pi'' = -2q + 2 = -2(4) + 2 = -6 < 0$, q = 4 maximises the profit [2]

<u>Another method</u>

MR = MC

$11 - q = q^2 - 3q + 3 \Rightarrow -q^2 + 2q + 8 = 0$

q = - 2 which economically not feasible, q = 4

(b) Determine the following integrals $\int \frac{x+1}{(x^2+2x+5)^2}dx$,

$t = x^2 + 2x + 5 \Rightarrow dt = 2(x+1)dx \Rightarrow (x+1)dx = dt/2$,

$$\int \frac{dt/2}{t^2} = \frac{1}{2}\int t^{-2}dt = \frac{-1}{2t} + C = \frac{-1}{2(x^2+2x+5)} + C$$ [5]

$$\int \frac{e^{-x} - e^{x}}{e^{x} + e^{-x}} dx$$, t = $e^x + e^{-x} \Rightarrow$ dt = $(e^x - e^{-x})$ dx =-($e^{-x} - e^x$)dx

$$\int \frac{-dt}{t} = -\ln|t| + C = -\ln(e^{x} + e^{-x}) + C$$

5

11. A firm faces a total cost function TC = $q^3 - 4q^2 + 12q$

(i) Determine the firm's average cost (AC) and marginal cost (MC) functions.

AC = $\frac{TC}{q}$ = $q^2 - 4q + 12$, MC = (TC)' = $3q^2 - 8q + 12$

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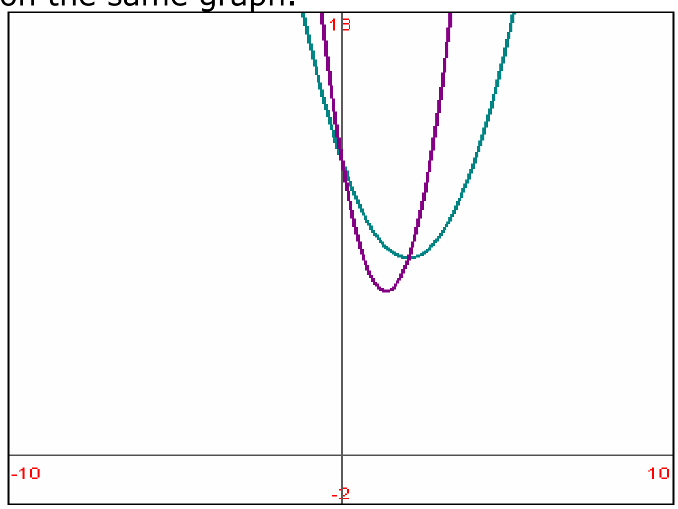
(ii) Sketch the average cost (AC) and the marginal cost(MC) on the same graph.



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(iii) If price is $ 15 , which level of output will a profit maximising firm choose?

8

TR = 15q

π = TR – TC = 15q – $q^3 + 4q^2 - 12q$ = – $q^3 + 4q^2 + 3q$

π'= $-3q^2 + 8q + 3 = 0 \Rightarrow$ q = -1/3 which is economically not feasible , q = 3

π'' = -6q + 8 = -6(3) + 8 = -10 < 0 $\Rightarrow$ q = 3 maximises the profit

END OF ANSWERS