International Institute for Technology and Management

Oct 7th ,2004

Tutoring Sheet #8

Unit 05a : Mathematics 1

Answers

1. Find the maxima and the minima of the following functions:

a. x^2

$f'(x) = 2x = 0 \Rightarrow x = 0$

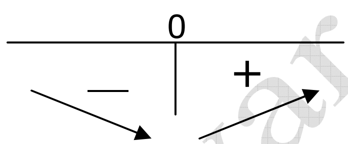
$f''(x) = 2 \Rightarrow f''(0) = 2 > 0 \Rightarrow x = 0$ minimizes f(x)

b. $2x^4 + 4$

$f'(x) = 8x^3 = 0 \Rightarrow x = 0$

$f''(x) = 24x^2 \Rightarrow f''(0) = 0 \Rightarrow$ no decision.

We have to study the sign of $f'(x) = 8x^3$



x = 0 minimizes f(x)

c. $x^3 - x$

$$f'(x) = 2x^2 - 1 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$$

$$f''(x) = 4x \Rightarrow f''\left(\frac{-1}{\sqrt{2}}\right) = \frac{-4}{\sqrt{2}} < 0$$

$$\Rightarrow x = \frac{-1}{\sqrt{2}} \text{ minimizes } f(x)\ ;\ f''\left(\frac{1}{\sqrt{2}}\right) = \frac{4}{\sqrt{2}} > 0$$

$$\Rightarrow x = \frac{1}{\sqrt{2}} \text{ maximizes } f(x).$$

d. $x^2 + 2x + 1$

$f'(x) = 2x+2 = 0 \Rightarrow x = -1$

$f''(x) = 2 \Rightarrow f''(-1) = 2 > 0 \Rightarrow x = -1$ minimizes f(x)

e. $2 + 4x - x^2$

$f'(x) = 4 - 2x = 0 \Rightarrow x = 2$

$f''(x) = -2 \Rightarrow f''(2) = -2 < 0 \Rightarrow x = 2$ maximizes f(x)

f. $2x^3 - 15x^2 + 36x + 4$

$f'(x) = 6x^2 - 30x + 36 = 0 \Rightarrow (6x - 12)(x - 3) = 0$

$\Rightarrow x = 2$ or $x = 3$

$f''(x) = 12x-30 \Rightarrow f''(2) = -6 < 0 \Rightarrow x = 2$ maximizes f(x)

$f''(3) = 6 > 0 \Rightarrow x = 3$ minimises f(x).

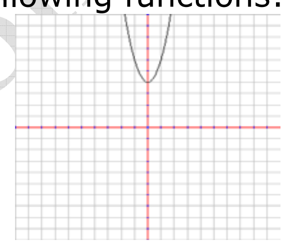
2. Find the maxima and the minima of the following functions:

a. $2x^2 + 4$

$f'(x) = 4x = 0 \Rightarrow x = 0$

$f''(x) = 4 \Rightarrow f''(0) = 4 > 0$

$\Rightarrow x = 0$ minimizes f(x)

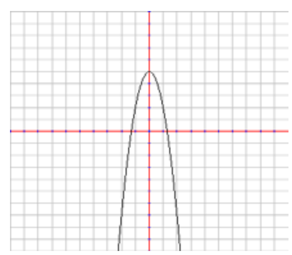


b. $5 - 3x^2$

$f'(x) = -6x = 0 \Rightarrow x = 0$

$f''(x) = -6 \Rightarrow f''(0) = -6 < 0$

$\Rightarrow x = 0$ maximizes f(x)



c. $2x^3 - 9x^2 - 24x + 10$

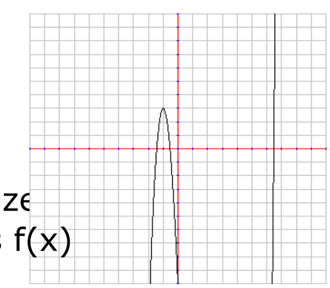
$f'(x) = 6x^2 - 18x - 24 = 0 \Rightarrow x = -1$

or $x = 4$

$f''(x) = 12x - 18$

$\Rightarrow f''(-1) = -30 < 0 \Rightarrow x = 0$ maximize

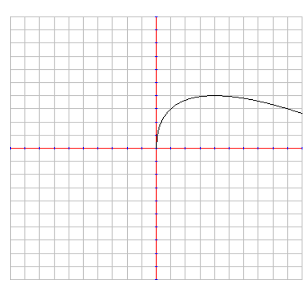
$f'(4) = 30 > 0 \Rightarrow x = 0$ minimizes f(x)



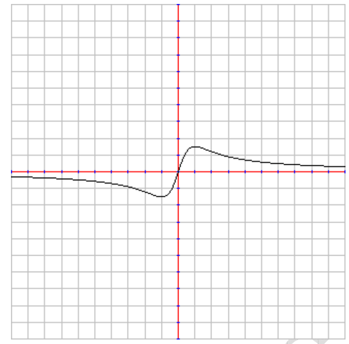
d. $4\sqrt{x} - x$

$f'(x) = \frac{4}{2\sqrt{x}} - 1 = 0 \Rightarrow x = 4$

$f''(x) = \frac{-1}{x\sqrt{x}}$, $f''(x) = \frac{-1}{8} < 0$ (Max)



e. $\frac{3x}{x^2+1}$



f′(x) = $\frac{3(x^2+1)-(2x)(3x)}{(x^2+1)^2}$

f′(x) = $\frac{-3x^2+3}{(x^2+1)^2}$ = 0 ⇒ $-3x^2 + 3 = 0$ ⇒ x = - 1 or x = 1

f″(x) = $\frac{(-6x)(x^2+1)^2-(-3x^2+3)(2x)(x^2+1)}{(x^2+1)^4}$

f″(-1) = $\frac{6(4)-0}{16} > 0$ ⇒ -1 minimizes f(x)

f″(1) = $\frac{-6(4)-0}{16} < 0$ ⇒ 1 maximizes f(x)

3. Suppose the demand and supply functions for a market are:

q^d = 1200 – 2p

q^s = 4p

Find the equilibrium price and quantity:
$q^d = q^s$
1200 – 2p = 4p ⇒ 6p = 1200 ⇒ p=200
q=4p=4(200) =800
The equilibrium Price and Quantity are:

P_0= 200 , q_0=800

4. Find all the local maxima and minima of the following functions,state whether each point is a maximum or minimum and find the value of the function at each point:

a. $y = x^2 - 4x + 2$
$y' = 2x - 4 = 0 \Rightarrow x = 2$
$y'' = 2 > 0 \Rightarrow x = 2$ minimizes the function.
To get the value of this minimum , substitute x = 2 in y :
$y = 2^2 - 4(2) + 2 = -4$

b. $y = x^3 - 3x^2$
$y' = 3x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0 \Rightarrow x = 0$ or $x = 2$
$y'' = 6x - 6$

For $x = 0$, $y'' = -6 < 0 \Rightarrow x = 0$ maximizes the function.
value of this maximum : $y = 0^3 - 3(0^2) = 0$

For $x = 2$, $y'' = 6 > 0 \Rightarrow x = 2$ minimizes the function.
value of this minimum : $y = 2^3 - 3(2^2) = -4$

c. $y = x + \frac{1}{x}$

$y' = 1 + \frac{-1}{x} = 1 - \frac{1}{x^2} \Rightarrow x^2 - 1 = 0 \Rightarrow x = -1$ or $x = 1$

$y'' = \frac{2}{x^3}$

For $x = -1$, $y'' = -2 < 0 \Rightarrow x = -1$ maximizes the function.

value of this maximum : $y = -1 + \frac{1}{-1} = -2$

For $x = 1$, $y'' = 2 > 0 \Rightarrow x = 1$ minimizes the function.

Value of this minimum : $y = 1 + \frac{1}{1} = 2$

d. $y = x^5$
$y' = 5x^4 = 0 \Rightarrow x = 0$
$y'' = 20x^3$
For $x = 0$, $y'' = 0 \Rightarrow$ Second Derivative Test *Fails*.
We need to study the sign of the *First* derivative:
$y' = 5x^4 > 0 \ \forall x$: + 0 +
$\Rightarrow$ No maximum or minimum at this inflexion point.

5. The average cost function of a firm is :

$$ac = 15 - 6q + q^2 + \frac{1}{q}$$

where q is the level of output .Derive the total cost and the marginal cost functions and sketch the average and marginal cost curves in the same diagram:

The total Cost : **TC = ac** × **q** = **q**$(15 - 6q + q^2 + \frac{1}{q})$

$\Rightarrow$ TC = $15q - 6q^2 + q^3 + 1$

The *Marginal Cost* function is the derivative of the cost function:

MC = $15 - 12q + 3q^2$

If the firm can sell as many units as it wishes at the price of 6 , What quantity will it sell if it is to maximize profits:

Total Revenue : TR = 6q

The profit function: $\prod$ = *total revenue – total cost*
=TR – TC=$6q - (15q - 6q^2 + q^3 + 1)$
$\prod = -q^3 + 6q^2 - 9q - 1$

For the profit to be maximum ,its derivative = 0
$\prod' = -3q^2 + 12q - 9 = 0 \Rightarrow q^2 - 4q + 3 = 0 \Rightarrow q = 1$ or $q = 3$

$\prod'' = -6q + 12$

For q = 1 , $\prod'' = 6 > 0 \Rightarrow q = 1$ minimizes the profit.

For q = 3 , $\prod'' = -6 < 0 \Rightarrow q = 3$ maximizes the profit.

The profit maximizing output is 3 .

What profit does it make at this output? Comment.

The profit is : PR = $-q^3 + 6q^2 - 9q - 1 = -(3)^3 + 6(3)^2 - 9(3) - 1 = -1$

The firm is running at a Loss.

6. Find the maximum value of the following functions(show it's maximum):

a. a. f(x) = $(1+x)e^{\frac{-x}{2}}$ of the form u.v

u = 1 + x $\Rightarrow$ u′ = 1 ; v = $e^{\frac{-x}{2}}$ $\Rightarrow$ v′ = $\frac{-1}{2}e^{\frac{-x}{2}}$

f′(x) = u′v + v′u = (1) $e^{\frac{-x}{2}}$ + $\frac{-1}{2}e^{\frac{-x}{2}}$ (1+x)

f′(x) = $e^{\frac{-x}{2}}\left(1-\frac{1+x}{2}\right)$ = $e^{\frac{-x}{2}}\left(\frac{1-x}{2}\right)$ = 0 $\Rightarrow$ x = 1

To verify it is a maximum ,use second derivative test:

f′(x) = $e^{\frac{-x}{2}}\left(\frac{1-x}{2}\right)$ $\Rightarrow$ f″(x) = $\frac{-1}{2}e^{\frac{-x}{2}}\left(\frac{1-x}{2}\right)+\frac{-1}{2}e^{\frac{-x}{2}}$

$\Rightarrow$ f″(1) = 0 - $\frac{1}{2}e^{\frac{-1}{2}}$ < 0 $\Rightarrow$ x = 1 maximizes f(x) .

To find the maximum ,substitute x = 1 in f(x)

f(1) = (1+1) $e^{\frac{-1}{2}}$ =2$e^{\frac{-1}{2}}$ = $\frac{2}{\sqrt{e}}$

b. f(x) = x - x ln x

f′(x) = 1 – [(1)(lnx) + (x)($\frac{1}{x}$)] = 1 – (lnx +1) = -lnx = 0

-lnx = 0 $\Rightarrow$ lnx = 0 $\Rightarrow$ x = e^0 = 1**(Recall:lnx=a $\Rightarrow$ x = e^a)**

To verify it is a maximum ,use second derivative test:

f″(x) = - $\frac{1}{x}$ $\Rightarrow$ f″(1) = -1 < 0 $\Rightarrow$ x = 1 maximizes f(x)

To find the maximum ,substitute x = 1 in f(x)

f(1) = 1 – (1)(ln1) = 1 – 0 = 1

7. Find the minimum value of the following functions(show it's minimum) :

a. f(x) = $e^{\sqrt{x}}-2\sqrt{x}$; $e^{\sqrt{x}}$ is of the form e^U;its derivative is

U′ e^U ; the derivative of $\sqrt{x}$ is $\frac{1}{2\sqrt{x}}$

$$f'(x) = \frac{1}{2\sqrt{x}} e^{\sqrt{x}} - 2\left(\frac{1}{2\sqrt{x}}\right) = \frac{1}{2\sqrt{x}} e^{\sqrt{x}} - \frac{1}{\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}}\left(½\, e^{\sqrt{x}} - 1\right) = 0 \Rightarrow ½\, e^{\sqrt{x}} - 1 = 0$$

$e^{\sqrt{x}} = 2 \Rightarrow \sqrt{x} = \ln 2$ **(Recall: $e^x=a \Rightarrow x = \ln a$)**

$x = (\ln 2)^2$

To verify it is a minimum ,use second derivative test:

$f'(x) = \frac{1}{\sqrt{x}}\left(½\, e^{\sqrt{x}} - 1\right)$ is of the form u.v

$u = \frac{1}{\sqrt{x}} = x^{-½} \Rightarrow u' = -½\, x^{-3/2}$

$v = ½\, e^{\sqrt{x}} - 1 \Rightarrow v' = ½\left(\frac{1}{2\sqrt{x}} e^{\sqrt{x}}\right) = \frac{1}{4\sqrt{x}} e^{\sqrt{x}}$

$f''(x) = u'v + v'u = \frac{-1}{4}\left(e^{\sqrt{x}} - 1\right) x^{-3/2} + \left(\frac{1}{4\sqrt{x}} e^{\sqrt{x}}\right)\left(\frac{1}{\sqrt{x}}\right)$

$f''(x) = \frac{-1}{4x^{\frac{3}{2}}}\left(e^{\sqrt{x}} - 1\right) + \frac{e^{\sqrt{x}}}{4x} > 0$

$e^{\sqrt{x}} = e^{\sqrt{(\ln 2)^2}} = e^{\ln 2} = 2$ **(Recall: $e^{\ln a}=a$)**

The value of the minimum : $f(\ln^2 2) = 2 - 2\ln 2$

b. $f(x) = x^2 - \ln(\sqrt{2}\, x)$

f′(x) = $2x - \frac{\sqrt{2}}{\sqrt{2}x} = 2x - \frac{1}{x}$

$f'(x) = 0 \Rightarrow 2x - \frac{1}{x} = 0 \Rightarrow 2x = \frac{1}{x} \Rightarrow 2x^2 = 1$

$x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

$f''(x) = 2 + \frac{1}{x^2} > 0 \ \forall x$

$x = \pm\frac{1}{\sqrt{2}}$ both minimize f

> F (x) = ln (cx)
>
> => $F'(x) = \frac{\mathbf{1}}{\boldsymbol{cx}} \times \boldsymbol{c} = \frac{\mathbf{1}}{\boldsymbol{x}}$
>
> Example: f(x) = ln 3x
>
> => $f'(x) = \frac{\mathbf{1}}{\boldsymbol{x}}$

8. A profit maximizing firm has the total cost function :

$$C = \frac{1}{3}q^3 - q^2 + 3q$$

and faces the demand schedule : $q = 30 - P$
where C and P are in £ 's.

Calculate the output of the firm which maximizes the profit.

The Net profit function:

$\prod$ = *total revenue – total cost*

$\prod$ = qp – C with q = 30 – P $\Rightarrow$ P = 30 – q

$\prod$ = q(30-q) - $\left(\frac{1}{3}q^3 - q^2 + 3q\right)$

$\prod$ = 27q - $\frac{1}{3}q^3$

$\prod$ **' = 27 – q² = 0** $\Rightarrow$ q = - $\sqrt{27} = -3\sqrt{3}$ or q = $\sqrt{27} = 3\sqrt{3}$,
q >0 $\Rightarrow$ q = $3\sqrt{3}$
$\prod$ **'' = -2q = -2(**$3\sqrt{3}$**) = -6**$\sqrt{3}$ **< 0** $\Rightarrow$ q = $3\sqrt{3}$ maximizes the Profit.