International Institute
for Technology and Management

Unit 05a : Mathematics 1 **Handout #18**

Partial Derivatives IV

Study Guide pp 89-93 ;Anthony & Biggs pp 253-259

Topic	Interpretation
Constrained Optimization: Suppose f(x,y) has to be minimized or maximized subject to the constraint g(x,y) = 0 Use the Lagrange Multipliers: 1.)Set: $L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$ 2.)Find x and y as solutions of: $\frac{\partial L}{\partial x} = 0$, $\frac{\partial L}{\partial y} = 0$, g(x,y)= 0 <u>Example1:</u> Use the Lagrange multiplier to find the values of x and y which maximizes $160x- 3x^2 -2xy -2y^2 +120y -18$ subject to the constraint x + y=34 *NB :* If no constraint is imposed The maximum is at (20,20) $f_1= 160- 6x - 2y = 0$ $f_2= -2x -4y +120 = 0$ Solving simultaneously,x = 20 y = 20 ;$f_{11} = -6, f_{22}= -4; f_{12} =-2$ $\Rightarrow (f_{11})(f_{22}) - f_{12}^2 =20 > 0$ Since $f_{11} = -6 < 0 \Rightarrow$ (20,20) maximizes f . **Applications of Constrained Optimization:** **1. Utility Function: u(x,y) :** For consuming two goods X and Y ,u(x,y) enables deciding between two bundles,i.e. ranking bundles.	$f(x,y) = 160x- 3x^2 -2xy -2y^2 +120y -18;$ $g(x,y) = x + y - 34 = 0$ $L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$ $L = 160x- 3x^2 -2xy -2y^2 +120y -18 - \lambda(x+ y - 34)$ $L = (160-\lambda)x-3x^2 -2xy -2y^2+(120-\lambda)y -18 + 34\lambda$ $\frac{\partial L}{\partial x} = 160-\lambda -6x -2y = 0$ -------(1) $\frac{\partial L}{\partial y} = -2x -4y + 120 - \lambda = 0$ ----(2) $g(x,y) = x + y - 34 = 0$ ---------(3) Eliminate λ from (1) & (2) : (1) $\lambda = 160 - 6x -2y$ (2) $\lambda = -2x -4y +120$; $\lambda = \lambda$ $160 - 6x -2y = -2x -4y +120$ $4x+2y+40=0 \Rightarrow 2x+y+20=0$ -----(4) (3) & (4) : x + y - 34 = 0 ; 2x + y + 20 = 0 x = 18 , y = 16 The constrained maximum f(18,16) = 2722 <u>Example 2:</u> u (10, 4) **>** u (9, 6) means that the consumer prefers to the bundle consisting of 10 of X & 4 of Y rather than 9 of X and 6 0f Y. <u>Example 3:</u> The prices of two goods X & Y are $P_x = 2$ and $P_y = 5$. The utility function $u(x,y) = x^{1/3}y^{1/2}$ and the income M = 40.Maximize the utility function subject to the budget constraint. The budget constraint :

Budget constraint:
Consumers normally seek the best bundle (highest utility-giving) they can afford. If the budget is **M** and the prices of X and Y are respectively P_x and P_y ,then consumers can only afford bundles (x,y) satisfying:
$xP_x + yP_y \leq M$.
i.e. consumers tend to maximize u(x,y) subject to the budget constraint $xP_x + yP_y = M$

2. Production Function : q(*k*,*l*)
q : production (quantity)
***k*: Capital ; *l* = Labour**

In order to produce a given quantity Q ,a firm usually chooses a combination of cost units of capital and labour such that the Cost is minimum.
If the cost of a unit of capital is *v* and the cost of a unit of labour (wage) is *w* then the cost is found by solving the following problem:
minimize $vk + wl$
subject to $q(k,l) = Q$
This is again a standard Constrained Optimization problem.

Production Maximization:
The problem here is to maximize Q(x,y) subject to the constraint that the company spends no more than a certain amount.

$xP_x + yP_y = M$; $2x + 5y = 40$
the problem becomes a standard *constrained optimization*:
Maximize $u(x,y) = x^{1/3}y^{1/2}$ subject to the constraint $2x + 5y = 40$.
Follow the steps mentioned in **Example1**
$x = 8$, $y = 24/5$.

Example 4: A firm's weekly output is given by the production function $q(k,l) = k^{3/4}\, l^{1/4}$.The unit costs for capital and labour are $v = 1$ and $w = 5$ per week Find the minimum cost of producing a weekly output of 5000 and the corresponding values of *k* and *l* .
The total cost : $vk + wl = k + 5l$
The problem to be solved is the constrained optimization:
Minimize $k + 5l$ subject to $k^{3/4}\, l^{1/4} = 5000$
Follow the steps mentioned in **Example1**
$k = 5000(15)^{1/4}$, $l = 5000(15)^{-3/4}$
Minimum cost = $k + 5l \approx 13120$

Example 5: A firm manufactures a good from two raw materials X and Y.The quantity of its good which is produced from x units of X and y units of Y is given by the quantity $Q(x,y) = x^{1/4}y^{3/4}$
If the firm spends no more than $1280 each week on the raw material,what is the maximum possible weekly production Given that one unit of X costs $16 and one unit of Y costs $1
The problem here is to :
Maximize $Q(x,y) = x^{1/4}y^{3/4}$ subject to the constraint $16x + y = 1280$
Follow the steps mentioned in **Example1**
$x = 20$; $y = 960$;Maximum production: $Q(20,960) = (20)^{1/4}(960)^{3/4}$