Assignment 5 $\boxed{\text{Solution}}$ Group A Feb. 16 09

1. a) $f(x,y) = x^2y + y^2$; $x = 3t^2+3 \Rightarrow \frac{dx}{dt} = 6t$

$y = t^3-7 \Rightarrow \frac{dy}{dt} = 3t^2$; $f_x = 2xy$; $f_y = x^2+2y$

$$F'(t) = \frac{\partial f}{\partial x} \times \frac{dx}{dt} + \frac{\partial f}{\partial y} \times \frac{dy}{dt}$$

$$= (2xy)(6t) + (x^2+2y)(3t^2)$$

$$= 2(3t^2+3)(t^3-7) + [(3t^2+3)^2 + 2(t^3-7)][3t^2]$$

$$\therefore F'(2) = 2(3(2)^2+3)(2^3-7) + [(3(2)^2+3)^2 + 2(2^3-7)][3(2)^2]$$

$$= 2(15)(1) + [225+2][12] = 2754.$$

b) $x^{\sqrt{y}} = 7$, $f_x = (\sqrt{y})\, x^{\sqrt{y}-1}$ ($\sqrt{y}$ here is constant)

$f_y = (x^{\sqrt{y}})(\ln x)\left(\frac{1}{2\sqrt{y}}\right)$ (here x is constant)

we applied: a^u, its derivative is $(a^u)(\ln a)(u')$.

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = -\frac{\sqrt{y}\, x^{\sqrt{y}-1}}{x^{\sqrt{y}} \ln x \left(\frac{1}{2\sqrt{y}}\right)} = \frac{-2(\sqrt{y})(\sqrt{y})\, x^{\sqrt{y}-1}}{x^{\sqrt{y}} \ln x}$$

$$\frac{dy}{dx} = \frac{-2y\, x^{\sqrt{y}-1-\sqrt{y}}}{\ln x} = \frac{-2y\, x^{-1}}{\ln x} = \frac{-2y}{x \ln x}.$$

2. a) $f(x,y) = x^2 - 2x - y^3 + y^2 + 8$

$f_x = 2x - 2 = 0 \Rightarrow x = 1.$

$f_y = -3y^2 + 2y = 0 \Rightarrow y(-3y+2) = 0$

$\Rightarrow y = 0$ or $y = \frac{2}{3}$.

Two critical points: $(1,0)$ and $(1,\frac{2}{3})$.

Existence: $f_{xx} = 2$; $f_{yy} = -6y+2$; $f_{xy} = 0$.

at $(1,0)$: $(f_{xx})(f_{yy}) - f_{xy}^2 = (2)(2) - 0 = 4 > 0$,

Yes, there is a critical point, since $f_{xx} = 2 > 0$

$\Rightarrow (1,0)$ minimises f.

at $(1, 2/3)$: $(f_{xx})(f_{yy}) - f_{xy}^2 = (2)(-2) - 0^2 = -4 < 0$

$\Rightarrow (1, 2/3)$ Saddle pt.

2. b) $f(x,y) = ye^{2y} \cdot x^{-a}$

$f_x = -a x^{-a-1} \cdot ye^{2y}$; $f_y = x^{-a}(1+2y)e^{2y}$.

$f_{xx} = -a(-a-1)x^{-a-2} \cdot ye^{2y} = a(a+1)x^{-a-2} \cdot ye^{2y}$.

$f_{yy} = x^{-a}\Big[\underbrace{2e^{2y}}_{u'v} \underset{+}{+} \underbrace{2(1+2y)e^{2y}}_{v'u}\Big] = 4x^{-a}(1+y)e^{2y}$.

$yx^2 f_{xx} - 3y f_{yy} + 12f = 0$

$\Rightarrow yx^2 (a)(a+1)x^{-a-2} \cdot ye^{2y} - 3y\left[4x^{-a}(1+y)e^{2y}\right] + 12ye^{2y}x^{-a} = 0$

$\Rightarrow x^{-a} y e^{2y}\left[\cancel{y}\cancel{x^2}(a)(a+1)\cancel{x^{-2}} - 12(1+y) + 12\right] = 0$.

$\Rightarrow (a^2+a)y - 12y = 0$ since $x, y > 0$ $x^{-a} y e^{2y} \neq 0$

$\Rightarrow a^2 + a - 12 = 0 \Rightarrow a = -4$ or $a = 3$.

3. $L = f - \lambda g = (x^{-2} + y^{-2})^{-1/2} - \lambda(x + y - \sqrt{2})$.

$L_x = -\frac{1}{2}(x^{-2} + y^{-2})^{-3/2} \cdot (-2x^{-3}) - \lambda = 0$

$\Rightarrow x^{-3}(x^{-2} + y^{-2})^{-3/2} - \lambda = 0 \Rightarrow \lambda = x^{-3}(x^{-2} + y^{-2})^{-3/2}$.

$L_y = -\frac{1}{2}(x^{-2} + y^{-2})^{-3/2} \cdot (-2y^{-3}) - \lambda = 0 \Rightarrow \lambda = y^{-3}(x^{-2} + y^{-2})^{-3/2}$

$\lambda = \lambda \Rightarrow x^{-3} = y^{-3} \Rightarrow x^{-1} = y^{-1} \Rightarrow x = y$.

using this with $x + y = \sqrt{2} \Rightarrow 2x = \sqrt{2}$

i.e. $x = \frac{\sqrt{2}}{2}$ & $y = \frac{\sqrt{2}}{2}$.

Maximum Value : $\left(\frac{1}{x^2} + \frac{1}{y^2}\right)^{-1/2} = \left(\frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2} + \frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2}\right)^{-1/2}$

$= (2+2)^{-1/2} = 4^{-1/2} = \frac{1}{4^{1/2}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$ //

4. $f(x,y) = \overset{v}{y}\,\overset{u \text{ in } f_y}{\ln(yx^{-1})} + \underset{u}{x}\,\underset{v}{e^{xy^{-1}}}$.

$f_x = y\left[\dfrac{-yx^{-2}}{yx^{-1}}\right] + (1)(e^{xy^{-1}}) + y^{-1}x e^{xy^{-1}}$

$f_x = -yx^{-1} + (1 + y^{-1}x)e^{xy^{-1}} = \dfrac{-y}{x} + \left(1 + \dfrac{x}{y}\right)e^{x/y}$.

$f_y = (y)\left[\dfrac{x^{-1}}{yx^{-1}}\right] + (1)\ln(yx^{-1}) + x(-xy^{-2})e^{xy^{-1}}$

$= 1 + \ln\left(\dfrac{y}{x}\right) - \dfrac{x^2}{y^2}e^{x/y}$.

$xf_x + yf_y = x\left[\dfrac{-y}{x} + e^{x/y} + \dfrac{x}{y}e^{x/y}\right] + y\left[1 + \ln\left(\dfrac{y}{x}\right) - \dfrac{x^2}{y^2}e^{x/y}\right]$

$= \cancel{-y} + xe^{x/y} + \cancel{\dfrac{x^2}{y}e^{x/y}} \cancel{+y} + y\ln\left(\dfrac{y}{x}\right) - \cancel{\dfrac{x^2}{y}e^{x/y}}$.

$= y\ln\left(\dfrac{y}{x}\right) + xe^{x/y} = f$ //

5. $x = 50 - \frac{1}{2}P_x \Rightarrow P_x = 100 - 2x$.

$y = 240 - 2P_y \Rightarrow P_y = 120 - \frac{1}{2}P_y$

$TR = xP_x + yP_y = x(100 - 2x) + y(120 - \frac{1}{2}y)$

$= -2x^2 + 100x + \frac{1}{2}y^2 + 120y$.

$\pi = TR - TC$

$\pi = -2x^2 + 100x - \frac{1}{2}y^2 + 120y - x^2 - 2xy - y^2 - 10$

$\pi = -3x^2 + 100x - \frac{3}{2}y^2 + 120y - 2xy - 10$.

$$\left.\begin{aligned} \pi_x &= -6x + 100 - 2y = 0 \\ \pi_y &= -3y + 120 - 2x = 0 \end{aligned}\right\} \Rightarrow x = \frac{30}{7};\ y = +\frac{260}{7}.$$

$\pi_{xx} = -6,\ \pi_{yy} = -3;\ \pi_{xy} = -2$

$(\pi_{xx})(\pi_{yy}) - \pi_{xy}^2 = 18 - (-2)^2 = 14 > 0$.

Since $\pi_{xx} = -6 < 0 \Rightarrow \left(\frac{30}{7}, \frac{260}{7}\right)$ maximises the profit.

5. b) $u(x,y) = 3\ln x_1 + \ln x_2$.

maximise $3\ln x_1 + \ln x_2$
subject to $p_1 x_1 + p_2 x_2 = M$.

$L = 3\ln x_1 + \ln x_2 - \lambda(p_1 x_1 + p_2 x_2 - M)$

$L_{x_1} = \frac{3}{x_1} - \lambda_1 p_1 = 0 \Rightarrow \lambda_* = \frac{3}{p_1 x_1}$.

$L_{x_2} = \frac{1}{x_2} - \lambda p_2 = 0 \Rightarrow \lambda = \frac{1}{p_2 x_2}$.

$\lambda = \lambda \Rightarrow \frac{3}{p_1 x_1} = \frac{1}{p_2 x_2} \Rightarrow p_1 x_1 - 3p_2 x_2 = 0$

use this with $p_1 x_1 + p_2 x_2 = M$.

$\left\{\begin{array}{l} p_1 x_1 - 3p_2 x_2 = 0 \\ \bar{p}_1 x_1 \mp p_2 x_2 = \bar{M} \end{array}\right\} \Rightarrow -4p_2 x_2 = -M$

$\Rightarrow x_2 = \frac{M}{4p_2}$; $p_1 x_1 - 3p_2 x_2 = 0 \Rightarrow p_1 x_1 = 3p_2 x_2$

$\Rightarrow x_1 = \frac{3p_2}{p_1} x_2 = \frac{3p_2}{p_1}\left(\frac{M}{4p_2}\right) = \frac{3M}{4p_1}$.

$(x_1^*, x_2^*) = \left(\frac{3M}{4p_1} ; \frac{M}{4p_2}\right)$.

END of Solution