Examination Papers
Examiners' Reports

Examination papers and
Examiners' reports

Mathematics 1

Economics, Management, Finance
and the Social Sciences

2003

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Examiner's report 2003

Zone A

Exam technique: general remarks

We start by emphasising that candidates should *always* include their working. This means two things. First, they should not simply write down the answer in the exam script, but explain the method by which it is obtained. Secondly, they should include rough working. The examiners want you to get the right answers, of course, but it is more important that you prove you know what you are doing: that is what is really being examined.

We also stress that if a student has not completely solved a problem, they may still be awarded marks for a partial, incomplete, or slightly wrong, solution; but, if they have written down a wrong answer and nothing else, no marks can be awarded.

Specific comments on questions

1. The fact that q is given as a function of p suggests that it is natural to place p on the horizontal and q on the vertical axis, but these could be interchanged. The supply and demand curves are quadratic. To sketch the supply curve accurately, you need to realise that it has a parabolic 'U' shape (if we plot q against p, as suggested above). Since the demand curve has a negative q^2 term, it is an upturned 'U' shape. An accurate sketch of each will need to indicate where the curves cross the axes, and will need to show also that the minimum of the supply and the maximum of the demand are to the left of the q axis. (The position of these can be obtained using differentiation.) The intercepts $(\sqrt{57}-7)/2$ of the supply curve and $(\sqrt{161}-1)/2$ of the demand curve on the p axis should also be shown. (And the values should be left like this—indeed, this has to be, since no calculators can be used). It is *never* adequate simply to determine a few points on the curve and then join them up: this is *plotting*, not *sketching*. (See the first paragraph on Page 19 of the Subject Guide.) To determine the equilibrium price, we solve $p^2+7p-2=-p^2-p+40$, which is equivalent to $2p^2+8p-42=0$, and has solutions $3, -7$, of which only 3 is economically meaningful. The equilibrium quantity is 28.

2. We find $f'(x)=c-1/x$. Solving $f'(x)=0$ gives $x=1/c$. Also, since $f''(x)=1/x^2>0$, $x=1/c$ gives a minimum. (We can also see this by examining how the sign of f' changes around $x=1/c$.) So, for all $x>0$, $f(x)\geq f(1/c)$, since $x=1/c$ gives a minimum. (We might note also that since $f(x)$ increases without bound as x increases, and since $x=1/c$ is the only local minimum, it is in fact a global minimum.) Thus, for all $x>0$,

$$f(x)\geq f(1/c)=c(1/c)-\ln(1/c)=1-\ln(1/c).$$

But $f(x)=cx-\ln x$, and so $cx-\ln x\geq 1-\ln(1/c)$, or

$$\ln x\leq cx+\ln(1/c)-1.$$

3. The easiest way to tackle this integral is by the substitution $u=\ln x$, on which it becomes $\int \frac{du}{u^2}$, and so the answer is $-1/u+c=-1/\ln x+c$.

4. We solve $\partial f/\partial x = 0 = \partial f/\partial y$ simultaneously. This gives $20x - 2y + 10 = 0 = -2x + 10y + 6$, with solution $x = -4/7$ and $y = -5/7$. The second derivative test must then be applied to prove that the critical point is a minimum: this requires noting *both* that $\partial^2 f/\partial x^2 > 0$ *and*

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} > \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2.$$

5. The problem is to minimise $3k + l$ subject to the constraint $4k^{1/2}l^{1/6} = Q$. (Some students had difficulty extracting this from the description in the problem.) The Lagrangean is therefore

$$L = 3k + l - \lambda(4k^{1/2}l^{1/6} - Q).$$

Then, the first-order conditions are

$$\begin{aligned}\frac{\partial L}{\partial k} = 3 - 2\lambda k^{-1/2}l^{1/6} &= 0\\ \frac{\partial L}{\partial l} = 1 - \frac{2}{3}\lambda k^{1/2}l^{-5/6} &= 0\\ 4k^{1/2}l^{1/6} &= Q.\end{aligned}$$

From the first two equations,

$$\lambda = \frac{3}{2}k^{1/2}l^{-1/6} = \frac{3}{2}k^{-1/2}l^{5/6},$$

which on simplification just becomes $k = l$. Then the third equation shows that $4k^{1/2}k^{1/6} = Q$ or $k^{2/3} = Q/4$ and hence $l = k = (Q/4)^{3/2} = Q^{3/2}/8$. (You do not need a calculator for this: $4^{3/2} = (4^{1/2})^3 = 2^3 = 8$.) The value of the minimum cost is therefore $3k + l = Q^{3/2}/2$. There is no need to prove that this minimises the cost: second-order tests for constrained optimisation are not included in this subject.

6. The system expressed in matrix form (a step some forgot) is

$$\begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 1 \\ -5 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 7 \end{pmatrix}.$$

(The system 'expressed in matrix form' is not the augmented matrix.) The standard matrix method approach is now to reduce the augmented matrix to reduced form. Here is one way. (There are others, equally valid.)

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 2 & 1 & 1 & 1 \\ -5 & -2 & 2 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 3 & 6 \\ 0 & -1 & -5 & -11 \\ 0 & 3 & 17 & 37 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 3 & 6 \\ 0 & 1 & 5 & 11 \\ 0 & 3 & 17 & 37 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 3 & 6 \\ 0 & 1 & 5 & 11 \\ 0 & 0 & 2 & 4 \end{pmatrix}$$

It follows, from this last matrix, that

$$\begin{aligned} x + y + 3z &= 6\\ y + 5z &= 11\\ 2z &= 4.\end{aligned}$$

So, $z = 2$, $y = 11 - 5(2) = 1$ and $x = 6 - 3(2) - 1(1) = -1$. Simply manipulating equations does not constitute a matrix method, which the question specifically asks for, and gets no credit. Some candidates used Cramer's rule. This is a technique not discussed in the Subject Guide, and the computations required are more difficult. We recommend the reduction technique, but Cramer's rule, carefully applied, gives the correct answer.

7. If y_N is the required amount, then we have

$$y_1 = (1.05)100 + 200, \quad y_2 = (1.05)y_1 + 200 = 100(1.05)^2 + 200(1.05) + 200,$$

and, in general,

$$\begin{aligned} y_N &= 100(1.05)^N + 200(1.05)^{N-1} + 200(1.05)^{N-2} + \cdots + 200(1.05) + 200 \\ &= 100(1.05)^N + 200\left(1 + (1.05) + (1.05)^2 + \cdots + (1.05)^{N-2} + (1.05)^{N-1}\right) \\ &= 100(1.05)^N + 200\frac{1-(1.05)^N}{1-(1.05)} \\ &= 100(1.05)^N + 4000\left((1.05)^N - 1\right), \end{aligned}$$

where we have used the formula for a geometric series.

8. (a) Because the firm is a monopoly, the selling price is, from the demand equation, $p = 20 - q$, and so the revenue is $q(20 - q) = 20q - q^2$. Now,

$$TC = \int MC dq = \int (3q^2 + 4)dq = q^3 + 4q + c,$$

and since the fixed cost is 10, we have $TC(0) = 10$ and hence $c = 10$. So the total cost is $q^3 + 4q + 10$. The profit is therefore $\Pi = -q^3 - q^2 + 16q - 10$. Solving $\Pi' = 0$ amounts to $-3q^2 - 2q + 16 = 0$. Solving this quadratic by the standard methods, we have $q = 2$ or $-8/3$, and clearly it is the positive solution that is economically meaningful. Then, $\Pi'' = -6q - 2$ and so $\Pi''(2) < 0$, showing that Π is maximised at 2. The maximum value of Π is $\Pi(2) = 10$.

(b) The total revenue is $xp_X + yp_Y = x(50 - 2x) + y(30 - y)$. So the profit is $\Pi = TR - TC = 50x - 2x^2 + 30y - 2y^2 - 2xy - 20$. The critical point is obtained by solving $\partial\Pi/\partial x = \partial\Pi/\partial y = 0$, and this leads to $x = 7$ and $y = 4$. Applying the second derivative test verifies that this gives a maximum of Π. The prices are $p_X = 36$ and $p_Y = 26$.

9. (a) In the first integral, the numerator is exactly one half of the derivative of the denominator, so the answer is $(1/2)\ln(x^2 + 2x + 8) + c$ (as can be seen explicitly by making the substitution $u = x^2 + 2x + 8$). For the second integral, partial fractions reveals that

$$\frac{x+3}{x^2+5x+4} = \frac{2/3}{x+1} + \frac{1/3}{x+4},$$

so the integral is $(2/3)\ln|x+1| + (1/3)\ln|x+4| + c$.

(b) Note that the question asks for the number of units sold after N weeks. This means the *total* sales up to the end of the Nth week, not just the sales during the Nth week. Consider the first model of sales growth, (i). Here, after N weeks, the total sales will be

$$1000 + (1000 + 100) + (1000 + 2(100)) + \cdots + (1000 + 100(N-1)).$$

By the formula for an arithmetic progression (with first term 1000, and common difference d, and with N terms), this simplifies to

$$\frac{N}{2}\left(2(1000) + (N-1)(100)\right) = 950N + 50N^2.$$

In case (ii), we instead have that the total sales are

$$1000 + 1000(1.02) + 1000(1.02)^2 + \cdots + 1000(1.02)^{N-1}$$

which, by the formula for a geometric progression (with first term 1000, common ratio 1.02, and N terms), is

$$1000\left(\frac{1-(1.02)^N}{1-1.02}\right) = 50000\left((1.02)^N - 1\right).$$

10. (a) Given the information in the question, we can see that

$$q(1) = 13 = a + b + c, \quad q(2) = 18 = a/4 + b/2 + 2c, \quad q(3) = 19 = a/9 + b/3 + 3c.$$

Multiplying the second and third equations by 4 and 9 (to get rid of fractions), we have the system

$$\begin{aligned} a + b + c &= 13 \\ a + 2b + 8c &= 72 \\ a + 3b + 27c &= 171. \end{aligned}$$

Solving by reduction, we have:

$$\begin{pmatrix} 1 & 1 & 1 & 13 \\ 1 & 2 & 8 & 72 \\ 1 & 3 & 27 & 171 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 13 \\ 0 & 1 & 7 & 59 \\ 0 & 2 & 26 & 158 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 13 \\ 0 & 1 & 7 & 59 \\ 0 & 1 & 13 & 79 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 13 \\ 0 & 1 & 7 & 59 \\ 0 & 0 & 6 & 20 \end{pmatrix}.$$

It follows that $c = 20/6 = 10/3$, $b = 59 - 7c = (177 - 70)/3 = 107/3$, and then $a = 13 - b - c = (39 - 10 - 107)/3 = -26$.

(b) By the chain rule,

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{x-y}{x+y}\right)^n &= n\left(\frac{x-y}{x+y}\right)^{n-1} \times \frac{\partial}{\partial x}\left(\frac{x-y}{x+y}\right) \\ &= n\left(\frac{x-y}{x+y}\right)^{n-1} \times \left(\frac{(x+y)-(x-y)}{(x+y)^2}\right) \\ &= 2ny\frac{(x-y)^n}{(x+y)^{n+1}}. \end{aligned}$$

Similarly,

$$\frac{\partial f}{\partial y} = -2nx\frac{(x-y)^n}{(x+y)^{n+1}}.$$

11. This is a constrained optimisation problem. We want to find the values of x_1 and x_2 that maximise the utility function $2\sqrt{x_1} + \sqrt{x_2}$ subject to the budget constraint $p_1x_1 + p_2x_2 = M$. The Lagrange multiplier method can then be used, with

$$L = 2\sqrt{x_1} + \sqrt{x_2} - \lambda\left(p_1x_1 + p_2x_2 - M\right).$$

The equations to be solved are

$$\begin{aligned} \frac{\partial L}{\partial x_1} = \frac{1}{\sqrt{x_1}} - \lambda p_1 &= 0 \\ \frac{\partial L}{\partial x_2} = \frac{1}{2\sqrt{x_2}} - \lambda p_2 &= 0 \\ p_1x_1 + p_2x_2 &= M. \end{aligned}$$

From the first two equations,

$$\lambda = \frac{1}{p_1\sqrt{x_1}} = \frac{1}{2p_2\sqrt{x_2}},$$

so $\sqrt{x_1} = 2(p_2/p_1)\sqrt{x_2}$ and hence $x_1 = 4p_2^2/p_1^2x_2$. Then the third equation tells us that

$$p_1\left(4\frac{p_2^2}{p_1^2}x_2\right) + p_2x_2 = M$$

and so the optimal x_1 and x_2 are given by

$$x_2^* = \frac{Mp_1}{(4p_2^2 + p_1p_2)}, \quad x_1^* = \frac{4Mp_2}{4p_1p_2 + p_1^2}.$$

The corresponding value of λ is

$$\lambda^* = \frac{1}{p_1\sqrt{x_1}} = \frac{1}{p_1}\sqrt{\frac{4p_1p_2 + p_1^2}{4Mp_2}} = \frac{1}{2\sqrt{M}}\sqrt{\frac{4p_2 + p_1}{p_1p_2}}.$$

For the last part, there is no need to calculate V: we can simply use the fact that $\partial V/\partial M = \lambda^*$ (and we have just calculated λ^*).

Examination paper for 2004

The format of the 2004 examination will be the same as that of the 2003 examination; namely seven compulsory questions in Section A and two questions to be chosen from four in Section B. Candidates should ensure that they have covered the bulk of the course in order to perform well in the examination: it is bad practice to concentrate only on a small range of major topics in the expectation that there will be lots of marks obtainable for questions on these topics. There are no formal options in this course: all students should study the full extent of the topics described in the syllabus and subject guide. In particular, since the whole syllabus is examinable, *any* topic could potentially appear in Section A.

Students are reminded that calculators are *not* permitted in the examination for this subject, under any circumstances. It is a good idea to prepare for this by attempting not to use your calculator as you study and revise this subject.