Examination Papers
Examiners' Reports

Examination papers and
Examiners' reports

Mathematics 2

Economics, Management, Finance
and the Social Sciences

2003

2790**05b**
9900**05b**
996D**05b**

Examiner's report 2003

Zone A

Exam technique: general remarks

We start by emphasising that candidates should *always* include their working. This means two things. First, they should not simply write down the answer in the exam script, but explain the method by which it is obtained. Secondly, they should include rough working. The examiners want you to get the right answers, of course, but it is more important that you prove you know what you are doing: that is what is really being examined.

We also stress that if a student has not completely solved a problem, they may still be awarded marks for a partial, incomplete, or slightly wrong, solution; but, if they have written down a wrong answer and nothing else, no marks can be awarded.

Specific comments on questions

1. To check homogeneity, we need to verify that $f(cx, cy) = c^2 f(x, y)$. The partial derivatives are

$$\frac{\partial f}{\partial x} = \frac{3x^2(x+y) - x^3}{(x+y)^2} + y\cos\left(\frac{x}{y}\right) - xy\frac{1}{y}\sin\left(\frac{x}{y}\right) = \frac{2x^3 + 3x^2y}{(x+y)^2} + y\cos\left(\frac{x}{y}\right) - x\sin\left(\frac{x}{y}\right),$$

$$\frac{\partial f}{\partial y} = \frac{-x^3}{(x+y)^2} + x\cos\left(\frac{x}{y}\right) + xy\frac{x}{y^2}\sin\left(\frac{x}{y}\right) = -\frac{x^3}{(x+y)^2} + x\cos\left(\frac{x}{y}\right) + \frac{x^2}{y}\sin\left(\frac{x}{y}\right).$$

Then,

$$\begin{aligned} x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= \frac{2x^4 + 3x^3y - x^3y}{(x+y)^2} + xy\cos\left(\frac{x}{y}\right) + xy\cos\left(\frac{x}{y}\right) - x^2\sin\left(\frac{x}{y}\right) + x^2\sin\left(\frac{x}{y}\right) \\ &= \frac{2x^3(x+y)}{(x+y)^2} + 2xy\cos\left(\frac{x}{y}\right) \\ &= \frac{2x^3}{x+y} + 2xy\cos\left(\frac{x}{y}\right) = 2f. \end{aligned}$$

2. To find the equilibrium price, we solve $bp - a = c - dp$, giving $p = (c+a)/(b+d)$. The corresponding quantity is $q = (bc - ad)/(b+d)$. The tax revenue is $R(T) = Tq^T$, where q^T is the new equilibrium quantity in the presence of tax. In order to calculate q^T, we first note than when the excise tax is imposed, the selling price at equilibrium, p^T, is such that

$$q^T = b(p^T - T) - a = c - dp^T.$$

(Note that we change the supply equation but not the demand equation. See the Subject Guide and texts for an explanation of this approach.) Solving for p^T, we obtain

$$p^T = \frac{c+a}{b+d} + \frac{bT}{b+d}.$$

Then

$$q^T = c - dp^T = \frac{bc-ad}{b+d} - \frac{bdT}{b+d},$$

so that

$$R(T) = \left(\frac{bc-ad}{b+d}\right)T - \left(\frac{bd}{b+d}\right)T^2.$$

Setting $R'(T) = 0$, we discover that there is only one critical point,

$$T_m = \frac{bc-ad}{2bd}.$$

The second derivative $R''(T) = (-2bd)/(b+d)$ is constant, and negative (since b and d are positive). Hence $R''(T_m) < 0$ and T_m is a maximum point. Therefore, T_m is the level of excise tax the government should impose. With this tax level, the production is

$$q = \frac{bc-ad}{b+d} = \frac{bd}{b+d}\frac{bc-ad}{2bd} = \frac{bc-ad}{2bd},$$

which is half the equilibrium quantity when there is no tax.

3. We need to solve $-\frac{p\,dq}{q\,dp} = 2p^2$. This is a separable equation, so, separating and integrating,

$$\ln q = \int \frac{dq}{q} = -\int 2p\,dp = -p^2 + c.$$

Taking exponentials, and renaming e^C as A,

$$q = \frac{A}{e^{p^2}}.$$

Given that $q(1) = 10$, we have $A/e = 10$ so $A = 10e$. Hence $q = 10e/e^{p^2} = 10/e^{p^2-1}$.

4. Cramer's rule and the inverse matrix method will not work here, since the matrix underlying the system has determinant 0. We instead need to use reduction. In matrix form, the system is

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & -3 & 2 \\ 3 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}.$$

Reducing the augmented matrix, we have (for example):

$$\begin{pmatrix} 2 & -1 & 1 & 4 \\ 1 & -3 & 2 & 4 \\ 3 & 1 & 0 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 2 & 4 \\ 2 & -1 & 1 & 4 \\ 3 & 1 & 0 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 2 & 4 \\ 0 & 5 & -3 & -4 \\ 0 & 10 & -6 & -8 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 2 & 4 \\ 0 & 5 & -3 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

From the last matrix, the system is equivalent to

$$\begin{aligned} x_1 - 3x_2 + 2x_3 &= 4 \\ 5x_2 - 3x_3 &= -4. \end{aligned}$$

Setting $x_3 = r$, any choice of number, will give a solution. Then, $x_2 = (1/5)(-4+3x_3) = -4/5+(3/5)r$ and $x_1 = 4-2x_3+3x_2 = 8/5-r/5$. So there are infinitely many solutions, and they are precisely

$$x_1 = \frac{8}{5} - \frac{r}{5},\ x_2 = -\frac{4}{5} + \frac{3}{5}r,\ x_3 = r$$

as r runs through all numbers.

5. There was quite a lot to do in this question, but it was all standard. (Incidentally, this question carries more credit than some other Section A questions. You should not assume that all questions carry the same credit. They are all compulsory in any case.) First, we need to find eigenvalues and eigenvectors of the matrix A. The characteristic polynomial is

$$p(\lambda) = \begin{vmatrix} -5-\lambda & -2 \\ 4 & 1-\lambda \end{vmatrix} = (-5-\lambda)(1-\lambda) + 8 = \lambda^2 + 4\lambda + 3.$$

The eigenvalues are the solutions to $p(\lambda) = 0$ and are therefore -1 and -3. To find an eigenvector corresponding to $\lambda = -1$, we must find a solution (other than the zero vector) to

$$\begin{pmatrix} -4 & -2 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This amounts to the single equation $4x + 2y = 0$. Taking $x = 1$ (for instance—any other non-zero choice will do), we obtain the eigenvector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. In a similar way, an eigenvector for $\lambda = -3$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, or any non-zero multiple of this. Now, if we take P to be

$$P = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix},$$

then $P^{-1}AP = D$ where

$$D = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}.$$

For the next part, the question explicitly asks ("Use your result...") that we use the result just obtained. So an alternative approach would not have been correct. The system of difference equations can be written as

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = A \begin{pmatrix} x_{t-1} \\ y_{t-1} \end{pmatrix}.$$

Following the approach outlined in the subject guide, let's introduce two related sequences X_t, Y_t by $\begin{pmatrix} x_t \\ y_t \end{pmatrix} = P\begin{pmatrix} X_t \\ Y_t \end{pmatrix}$. Then, the system of equations is equivalent to

$$P\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = A\begin{pmatrix} x_{t-1} \\ y_{t-1} \end{pmatrix} = AP\begin{pmatrix} X_{t-1} \\ Y_{t-1} \end{pmatrix}.$$

So,

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = P^{-1}AP\begin{pmatrix} X_{t-1} \\ Y_{t-1} \end{pmatrix} = D\begin{pmatrix} X_{t-1} \\ Y_{t-1} \end{pmatrix}.$$

So,

$$X_t = -X_{t-1}, \quad Y_t = -3Y_{t-1}$$

and hence, for some numbers A and B, $X_t = A(-1)^t$ and $Y_t = B(-3)^t$. We're interested in determining x_t and y_t. We have

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = P\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} A(-1)^t + B(-3)^t \\ -2A(-1)^t - B(-3)^t \end{pmatrix}.$$

Since $x_0 = 1$ and $y_0 = 2$, we must have $A + B = 1$ and $-2A - B = 2$, so $A = -3$, and $B = 4$. Finally, then, we have the solution:

$$x_t = -3(-1)^t + 4(-3)^t, \quad y_t = 6(-1)^t - 4(-3)^t.$$

6. The auxiliary equation, $z^2 - 6z + 9 = 0$, has the single solution $z = 3$. For a particular solution, substituting $f = ax + b$ shows that $a = 1$ and $b = -1$. So, for some constants A and B,

$$f(x) = x - 1 + Ae^{3x} + Bxe^{3x}.$$

The fact that $f(0) = 0$ means $-1 + A = 0$, so $A = 1$; and the fact that $f(1) = 0$ means $Ae^3 + Be^3 = 0$, so $B = -A = -1$. Therefore, $f(x) = x - 1 + e^{3x} - xe^{3x}$. (Note that we are not given $f(0)$ and $f'(0)$: we are given $f(0)$ and $f(1)$. Read the question carefully and do not assume it will be like every other such question you have seen. In fact, this is easier, since one does not have to calculate $f'(x)$.)

7. (a) There are various approaches. One is to directly use Taylor's theorem, calculating $f'(0), f''(0), f^{(3)}(0)$ and $f^{(4)}(0)$ and using

$$f(x) \simeq f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4.$$

These successive derivatives do, however, get harder to compute. An easier approach is to use the standard result (which you may assume) concerning the series for e^y, which is

$$e^y \simeq 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24} + \cdots.$$

Taking $y = x^2 - x$, we obtain

$$e^{x^2-x} \simeq 1 + (x^2 - x) + \frac{1}{2}(x^2 - x)^2 + \frac{1}{6}(x^2 - x)^3 + \frac{1}{24}(x^2 - x)^4 + \cdots.$$

It's a lot easier at this stage if we take out the factor of x inside each bracket. This helps us determine what terms we need to worry about on expanding the powers. The calculation proceeds as follows, where all powers higher than x^5 are simply ignored:

$$\begin{aligned} e^{x^2-x} &\simeq 1 + x(x-1) + \frac{1}{2}x^2(x-1)^2 + \frac{1}{6}x^3(x-1)^3 + \frac{1}{24}x^4(x-1)^4 + \cdots \\ &= 1 + x^2 - x + \frac{1}{2}x^2(x^2-2x+1) + \frac{1}{6}x^3(3x-1) + \frac{1}{24}x^4 + \cdots \\ &= 1 - x + \frac{3}{2}x^2 - \frac{7}{6}x^3 + \frac{25}{24}x^4 + \cdots . \end{aligned}$$

(b) One method is to write the system in matrix form as

$$\begin{pmatrix} df/dt \\ dg/dt \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix},$$

and then use the diagonalisation method. A second approach is to reduce the two equations to a single second-order equation, and it is this method that we now explain. (You should, of course, get the same answer with the diagonalisation method! Try both for practice.) Differentiating the first equation and using the given information, we have

$$\begin{aligned} \frac{d^2f}{dt^2} &= 5\frac{df}{dt} - \frac{dg}{dt} \\ &= 5\frac{df}{dt} + 2f - 4g \\ &= 5\frac{df}{dt} + 2f - 4\left(5f - \frac{df}{dt}\right) \\ &= 9\frac{df}{dt} - 18f, \end{aligned}$$

So $\frac{d^2f}{dt^2} - 9\frac{df}{dt} + 18f = 0$. This has auxiliary equation $z^2 - 9z + 18 = 0$, with solutions 3 and 6. So, for some A, B, $f(t) = Ae^{3t} + Be^{6t}$. Then,

$$g(t) = 5f(t) - \frac{df}{dt} = 5Ae^{3t} + 5Be^{6t} - 3Ae^{3t} - 6Be^{6t} = 2Ae^{3t} - Be^{6t}.$$

Given $f(0) = 2$ and $g(0) = 1$, we have $A + B = 2$ and $2A - B = 1$, so $A = 1, B = 1$. Finally, therefore,

$$f(t) = e^{3t} + e^{6t}, \quad g(t) = 2e^{3t} - e^{6t}.$$

8. (a) There are at least two distinct approaches to finding the inverse: one is to use cofactors, and the other is by row operations. See the Subject Guide for details. The inverse of this matrix turns out to be

$$\begin{pmatrix} 3/4 & 1/2 & 0 \\ 1/4 & 1/2 & 0 \\ 1/4 & 1/2 & -1 \end{pmatrix}.$$

(b) The equation can be written as

$$\frac{dS}{dt} - \frac{1}{20}S = -t - 2$$

from which it is clear that it is a linear differential equation. The integrating factor is $\exp\left(\int -(1/20)dt\right) = e^{-t/20}$. So,

$$Se^{-t/20} = -\int (t+2)e^{-t/20}dt.$$

The integral can be done by parts, and we end up with

$$Se^{-t/20} = 20(t+2)e^{-t/20} + 400e^{-t/20} + c,$$

so

$$S = 20(t+2) + 400 + ce^{t/20} = 440 + 20t + ce^{t/20}.$$

The fact that $S(0) = P$ leads to $c = P - 440$. So

$$S = 440 + 20t + (P - 440)e^{t/20}.$$

When $P = 300$, $S(t) = 440 + 20t - 140e^{t/20}$ and $S'(t) = 20 - e^{t/20}$. The derivative is 0 when $t/20 = \ln(20/7)$, that is $t = t^* = 20\ln(20/7)$. Furthermore, $S''(t) = -(7/20)e^{t/20} < 0$, so t^* gives a maximum value.

9. The elasticity is given by

$$\varepsilon = -\frac{p}{q}\frac{dq}{dp} = -\frac{p}{q}\frac{18 - 5p^2}{(p^2+18)^4},$$

where we've omitted the details of the differentiation. Now, this is not yet the answer, because we need to find the elasticity in terms of p only. We must therefore eliminate q:

$$\varepsilon = -\frac{p}{q}\frac{18-5p^2}{(p^2+18)^4} = -\frac{p}{p/(p^2+18)^3}\frac{18-5p^2}{(p^2+18)^4} = \frac{5p^2-18}{p^2+18}.$$

Demand is inelastic if and only if $\varepsilon < 1$, which is equivalent to $5p^2 - 18 < p^2 + 18$, or $4p^2 < 36$. So we need $p^2 < 9$ which, given that prices cannot be negative, means $0 \le p < 3$.

(b) The difference equation has to be written in standard form before the usual techniques can be applied. This is

$$y_{t+1} - \frac{1}{c}y_t + \frac{(1-c)}{c}y_{t-1} = 0.$$

The auxiliary equation is $z^2 - (1/c)z + ((1-c)/c) = 0$ and this has solutions

$$\begin{aligned} \frac{1}{2}\left(\frac{1}{c} \pm \sqrt{\frac{1}{c^2} - 4\frac{(1-c)}{c}}\right) &= \frac{1}{2c}\left(1 \pm \sqrt{1 - 4c + 4c^2}\right) \\ &= \frac{1}{2c}\left(1 \pm \sqrt{(1-2c)^2}\right) \\ &= \frac{1}{2c}\left(1 \pm (1-2c)\right), \end{aligned}$$

so the two solutions are $(1-c)/c$ and 1. (Alternatively, the solutions can be found by noting that $z^2 - (1/c)z + (1-c)/c = (z - (1-c)/c)(z-1)$. It follows now that for some A, B,

$$y_t = A(1)^t + B\left(\frac{1-c}{c}\right)^t = A + B\left(\frac{1-c}{c}\right)^t.$$

10. (a) The Lagrangean is

$$L = x^4 + y^4 + x^4 - \lambda(x + 8y + 27z - 10)$$

and the first-order conditions are

$$\begin{aligned} \frac{\partial L}{\partial x} = 4x^3 - \lambda &= 0 \\ \frac{\partial L}{\partial y} = 4y^3 - 8\lambda &= 0 \\ \frac{\partial L}{\partial z} = 4z^3 - 27\lambda &= 0 \\ x + 8y + 27z &= 10. \end{aligned}$$

From the first three equations,

$$\lambda = 4x^3 = \frac{4y^3}{8} = \frac{4z^3}{27}.$$

So, $y^3 = 8x^3$ and $z^3 = 27x^3$, from which we see that $y = 2x$ and $z = 3x$. The final equation then becomes $x + 16x + 81x = 10$, so $x = 10/98 = 5/49$ and then $y = 2x = 10/49$, $z = 3x = 15/49$.

(b) The equation for p becomes

$$\frac{dp}{dt} = ((1-p) - p)^5 = (1-2p)^5.$$

This is separable, and we have:

$$\int \frac{dp}{(1-2p)^5} = \int dt,$$

so

$$\frac{1}{8}\frac{1}{(1-2p)^4} = t + c.$$

Given that $p(0) = 1/4$, $c = 2$. From

$$\frac{1}{(1-2p)^4} = 8(t+2) = 8t + 16,$$

we have

$$(1-2p) = \pm\left(\frac{1}{8t+16}\right)^{1/4}.$$

Given that $p(0) = 1/4$, we take the $+$ sign and end up with

$$p = \frac{1}{2} - \frac{1}{2}\frac{1}{(8t+16)^{1/4}},$$

which tends to $1/2$ as t tends to infinity.

Examination paper for 2004

The format of the 2004 examination will be the same as that of the 2003 examination; namely six compulsory questions in Section A and two questions to be chosen from 4 in Section B. Candidates should ensure that they have covered the bulk of the course in order to perform well in the examination: it is bad practice to concentrate only on a small range of major topics in the expectation that there will be lots of marks obtainable for questions on these topics. There are no formal options in this course: all students should study the full extent of the topics described in the syllabus and subject guide. In particular, since the whole syllabus is examinable, *any* topic could potentially appear in Section A.

Students are reminded that calculators are *not* permitted in the examination for this subject, under any circumstances. It is a good idea to prepare for this by attempting not to use your calculator as you study and revise this subject.