International Institute for
Technology and Management

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Unit 05b: Mathematics 2 – (MathA) Assignment – 1

1. The demand and the supply functions of a market are:

$$q^d = a - bp \quad \text{and} \quad q^s = c + dp$$

where a ,b ,c and d are positive real numbers.

a. Show that the equilibrium price and the quantity are :

$$p_0 = \frac{a-c}{b+d} \quad ; \quad q_0 = \frac{ad+bc}{b+d}$$ **(1 Mark)**

$q^d = q^s$: $a - bp_0 = c + dp_0 \Rightarrow bp_0 + dp_0 = a - c$

$(b + d)p_0 = a - c \Rightarrow p_0 = \dfrac{a-c}{b+d}$

Substituting in q^d : $q_0 = a - bp_0 = a - b\left(\dfrac{a-c}{b+d}\right)$

$$q_0 = a - \frac{ba-bc}{b+d} = \frac{ab+ad-ba+bc}{b+d} = \frac{ad+bc}{b+d}$$

b. If an excise tax of t is imposed ,find the new equilibrium price $\mathbf{p_1}$ and quantity $\mathbf{q_1}$ **(2 Marks)**

When an excise tax t is imposed , from the supplier's point of view ,the price becomes(p – t) ,the new supply function :

$q^s = c + d(p - t) = c + dp - dt$.

The demand function remains the same : $q^d = a - bp$

If (q_1 , p_1) the new equilibrium price and quantity ,

(q_1 , p_1) satisfy both the equations of the modified supply and the demand functions:

$c + dp_1 - dt = a - bp_1 \Rightarrow (b + d)p_1 = a - c + dt$

$$\Rightarrow p_1 = \frac{a-c}{b+d} + \left(\frac{d}{b+d}\right)t$$

Substituting p_1 in q^d : $q_1 = a - bp_1 = a - \dfrac{ba-bc}{b+d} - \dfrac{bd}{b+d}t$

$$q_1 = \frac{ab+ad-ba+bc}{b+d} - \frac{bd}{b+d}t = \frac{ad+bc}{b+d} - \frac{bd}{b+d}t$$

c. Find the tax revenue and the value of **t** that maximises the tax revenue. **(3 Marks)**

Tax Revenue : T = q_1 t = $\left(\frac{ad+bc}{b+d} - \frac{bd}{b+d}\,t\right)t$

$$= \frac{ad+bc}{b+d}\,t - \frac{bd}{b+d}\,t^2$$

$$T' = \frac{ad+bc}{b+d} - \frac{2bd}{b+d}\,t = 0 \Rightarrow \frac{ad+bc}{b+d} = \frac{2bd}{b+d}\,t$$

$$\Rightarrow ad + bc = 2bd\,t \Rightarrow t = \frac{ad+bc}{2bd}$$

$$T'' = -\frac{2bd}{b+d} \prec 0 \quad \text{i.e.} \quad t = \frac{ad+bc}{2bd} \text{ maximises } T.$$

d. Show that the tax that maximises the tax revenue reduces sales by exactly half.(Hint: show that $\mathbf{q_1 = ½\, q_0}$) **(4 Marks)**

The value of t that maximises the tax is $t = \frac{ad+bc}{2bd}$

$$\Rightarrow q_1 = \frac{ad+bc}{b+d} - \frac{bd}{b+d}\,t = \frac{ad+bc}{b+d} - \frac{bd}{b+d} \times \frac{ad+bc}{2bd}$$

$$= \frac{ad+bc}{b+d} - \frac{1}{b+d} \times \frac{ad+bc}{2} = \frac{ad+bc}{b+d} - \frac{ad+bc}{2(b+d)}$$

$$\Rightarrow q_1 = \frac{2(ad+bc)-(ad+bc)}{2(b+d)} = \frac{ad+bc}{2(b+d)} = \frac{1}{2}\left(\frac{ad+bc}{b+d}\right) = \frac{1}{2}q_0$$

That is, sales have been reduced exactly to half.

e. The demand and the supply sets for a market are respectively:

p – q = 2 and 2p – 3q = -1

Find the price and the quantity at equilibrium when a purchase tax of **t** is imposed. **(2 Marks)**

The new demand function : q = (p + t) – 2 = p + t – 2

The supply function remains the same : q = $\frac{2p+1}{3}$

Let q^t , p^t be the new equilibrium quantity and price

(q^t, p^t) satisfy the equations of both the demand and the supply functions.

Equating both equalities: $p^t + t - 2 = \dfrac{2p^t + 1}{3} \Rightarrow p^t = 7 - 3t$

Substituting this in one of the above equations:

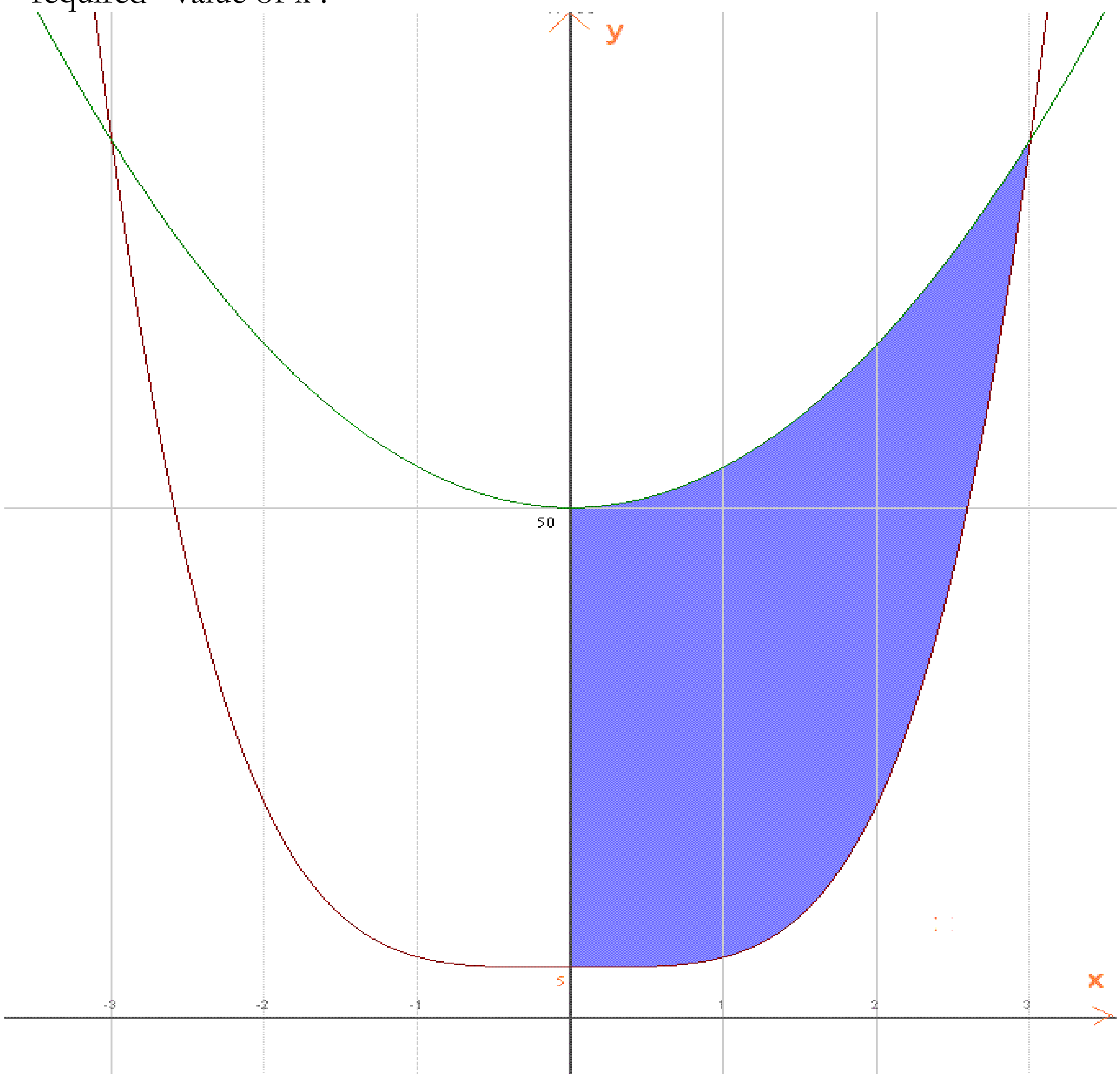
$q^t = p^t + t - 2 = 5 - 2t \Rightarrow (q^t, p^t) = (5 - 2t, 7 - 3t)$

2. Find the positive value of x at which the curves with equations $\mathbf{y = 50 + 4x^2}$ and $\mathbf{y = x^4 + 5}$ intersect. Produce a rough sketch of the curves. Find the area of the region with $x > 0$ enclosed by these curves and the y-axis **(8 Marks)**

Intersection : y = y $\Rightarrow$ $x^4 + 5 = 50 + 4x^2 \Rightarrow x^4 - 4x^2 - 45 = 0$

$$x^2 = \frac{4 \pm \sqrt{196}}{2} = \frac{4 \pm 14}{2}$$

Either $x^2 = -6$ i.e. No roots or $x^2 = 9 \Rightarrow \mathbf{x = \pm 3}$ i.e $x = 3$ is the required value of x .



$$\text{Area} = \left|\int_0^3 [(4x^2+50)-(x^4+5)]dx\right|$$

$$= \left| -\frac{x^5}{5}+\frac{4x^3}{3}+45x\Big|_0^3 \right| = \left| -\frac{3^5}{5}+\frac{4(3^3)}{3}+45(3)-0\right|$$

$$= \left| -\frac{243}{5}+171\right| = \frac{612}{5} \text{ square units.}$$

3. The point elasticity of demand for a good is given by $\varepsilon = 2$

Given that $\mathbf{q^D(5) = 4}$, find the demand function $\mathbf{q^D}$ **(p)** **(5 Marks)**

$$\varepsilon = \frac{-p}{q}\frac{dq}{dp} = 2 \Rightarrow \frac{dq}{q} = \frac{-2}{p}dp \text{ Integrating both sides}$$

$$\ln q = -2\ln p + C = \ln p^{-2} + \ln e^C = \ln\frac{e^C}{p^2} \Rightarrow q = \frac{e^C}{p^2} = \frac{k}{p^2}$$

$$q^D(5) = 4 \Rightarrow 4 = \frac{k}{5^2} \Rightarrow k = 100 \Rightarrow q = \frac{100}{p^2}$$

4. Suppose the demand function for a commodity is given by

$$q = \frac{p}{(p^2+8)^3}$$

Find the elasticity of demand in terms of ***p***. Determine the value of ***p*** for which the demand is elastic. **(5 Marks)**

The elasticity of demand is given by

$$\varepsilon = -\frac{p}{q}\frac{dq}{dp} = -\frac{p}{q}\frac{8-5p^2}{(p^2+8)^4},$$

where we've omitted the details of the differentiation. Now, this is not yet the answer, because we need to find the elasticity in terms of p only. We must therefore eliminate q:

$$\varepsilon = -\frac{p}{q}\frac{8-5p^2}{(p^2+8)^4} = -\frac{p}{p/(p^2+8)^3}\frac{8-5p^2}{(p^2+8)^4} = \frac{5p^2-8}{p^2+8}.$$

Demand is elastic if and only if $\varepsilon > 1$, which is equivalent to $5p^2 - 8 > p^2 + 8$, or $4p^2 > 16$. So (noting that $p \geq 0$) we need $p > 2$.

5. Expand as a power series, the functions: **(10 Marks)**

a. $f(x) = \ln\left(\dfrac{1+2x}{1+x}\right)$ in terms up to $\mathbf{x^4}$

$$f(x) \simeq f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4 + \frac{f^{(5)}(0)}{120}x^5.$$

These successive derivatives do, however, get harder to compute. An easier approach is note that $f(x) = \ln(1+2x) - \ln(1+x)$ and to use the standard result (which you may assume) concerning the series for $\ln(1+y)$, which is

$$\ln(1+y) \simeq y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} + \cdots$$

Taking $y = 2x$ and x, we obtain

$$\begin{aligned} f(x) &= \ln(1+2x) - \ln(1+x) \\ &\simeq \left(2x - \frac{1}{2}(2x)^2 + \frac{1}{3}(2x)^3 - \frac{1}{4}(2x)^4 + \frac{1}{5}(2x)^5\right) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}\right) + \\ &= 2x - 2x^2 + \frac{8}{3}x^3 - 4x^4 + \frac{32}{5}x^5 - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} + \cdots \\ &= x - \frac{3}{2}x^2 + \frac{7}{3}x^3 - \frac{15}{4}x^4 + \frac{31}{5}x^5 + \cdots. \end{aligned}$$

b. $f(x) = \dfrac{e^{2x}}{1-x}$ in terms up to $\mathbf{x^3}$

$$f(x) \simeq f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3.$$

However, an easier approach is to use the standard results (which you may assume) concerning the series for e^y and $(1-x)^{-1}$, which are

$$e^y \simeq 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \cdots$$

and

$$(1-x)^{-1} \simeq 1 + x + x^2 + x^3 + \cdots.$$

Then,

$$e^{2x}(1-x)^{-1} \simeq \left(1 + (2x) + \frac{(2x)^2}{2} + \frac{(2x)^3}{6}\right)\left(1 + x + x^2 + x^3\right),$$

which, after careful manipulation, and omitting powers of x higher than 3, becomes

$$1 + 3x + 5x^2 + \frac{19}{3}x^3.$$

c. $f(x) = \sin(x^2 - x)$ in terms up to $\mathbf{x^5}$

$$f(x) \simeq f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4 + \frac{f^{(5)}(0)}{120}x^5.$$

These successive derivatives do, however, get harder to compute. An easier approach is to use the standard result (which you may assume) concerning the series for $\sin y$, which is

$$\sin y \simeq y - \frac{y^3}{6} + \frac{y^5}{120} + \cdots.$$

Taking $y = x^2 - x$, we obtain

$$\sin(x^2 - x) \simeq (x^2 - x) - \frac{1}{6}(x^2 - x)^3 + \frac{1}{120}(x^2 - x)^5 \cdots.$$

It's a lot easier at this stage if we take out the factor of x inside each bracket. This helps us determine what terms we need to worry about on expanding the powers. The calculation proceeds as follows, where all powers that would result in terms higher than x^5 in power are simply ignored:

$$\begin{aligned}\sin(x^2 - x) &\simeq x^2 - x - \frac{1}{6}x^3(x-1)^3 + \frac{1}{120}x^5(x-1)^5 + \cdots \\ &= x^2 - x - \frac{1}{6}x^3(-3x^2 + 3x - 1) + \frac{1}{120}x^4(-1) + \cdots \\ &= -x + x^2 + \frac{1}{6}x^3 - \frac{1}{2}x^4 + \frac{59}{120}x^5 + \cdots.\end{aligned}$$

d. $f(x) = \sqrt{\dfrac{1+x}{1-x}}$ in terms up to $\mathbf{x^3}$

by taking a suitable value for x, find $\sqrt{11}$.

f(0) = 1

$$\sqrt{\frac{1+x}{1-x}} = \sqrt{\frac{2}{1-x} - 1} = \left(\frac{2}{x-1} - 1\right)^{1/2}$$

Or $$\sqrt{\frac{1+x}{1-x}} = \sqrt{\frac{(1+x)^2}{(1+x)(1-x)}} = (1+x)(1-x^2)^{-1/2}$$

$$\sqrt{\frac{1+x}{1-x}} = (1+x)(1 + \tfrac{1}{2}x^2 + \tfrac{3}{4}x^4 + \ldots..) = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3$$

Put $\sqrt{\dfrac{1+x}{1-x}} = \sqrt{11} \Rightarrow$ **x = 10/12 = 5/6**

$$\sqrt{11} = 1 + 5/6 + \tfrac{1}{2}(5/6)^2 + \tfrac{1}{2}(5/6)^3$$

6. The demand equation for a good is $\mathbf{p = 113 - q^2}$ and the supply equation is $\mathbf{p - q^2 - 2q = 1}$

Calculate the consumer surplus and the producer surplus at equilibrium.

(5 Marks)

We solve $113 - q^2 = 1 + q^2 + 2q$, which, in standard form, is $q^2 + q - 56 = 0$, with positive solution $q = 7$, which is therefore the equilibrium quantity. The corresponding price is $p = 113 - q^2 = 64$. To find the consumer and producer surplus, it might be useful to sketch roughly the graphs so that we know what areas to determine. The consumer surplus is

$$CS = \int_0^7 (113 - q^2)\, dq - 7(64) = 686/3.$$

The producer surplus is

$$PS = 7(64) - \int_0^7 (1 + q^2 + 2q)\, dq = 833/3.$$

7.

Suppose the supply and demand functions for a good are, respectively,

$$q^S(p) = 5p - 2, \quad q^D(p) = 12 - 2p.$$

Determine the equilibrium price and quantity. A percentage sales tax of $100r\%$ is imposed. (So, when a consumer buys one unit of the good at a price p, an amount rp is tax.) Find the new equilibrium price and quantity. Find also an expression for the amount of tax revenue.

(5 Marks)

(b) The equilibrium price is given by $5p - 2 = 12 - 2p$, so $p = 2$. The equilibrium quantity is 8. When a tax of the type described is imposed, the new equilibrium price is given by

$$5p(1 - r) - 2 = 12 - 2p.$$

(Note that we change the supply equation but not the demand equation. The reasoning is that, as far as the suppliers are concerned the effective price is the price minus the tax, which is $p(1 - r)$.) Solving for p, we obtain $p = 14/(7 - 5r)$. Then $q = 12 - 2p = (56 - 60r)/(7 - 5r)$. The tax revenue per unit is rp, so the total tax revenue is rpq, which is

$$r\frac{14}{(7 - 5r)}\frac{(56 - 60r)}{(7 - 5r)} = \frac{14r(56 - 60r)}{(7 - 5r)^2}.$$

<u>END of QUESTIONS</u>